Identities - Kick-off- Meeting
Bologna – November 19th 2019

Montpellier approach to interdisciplinarity and examples of modules and activities that can be carried out in the Summer schools:

Logical and linguistic approach
First insights

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*“Logic seems, contrary with what Wittgenstein thought, an indispensable mean, not of ‘founding’ but of understanding mathematical activity. That means for a part to explore the relation from implicit to explicit in a theory (…). An essential part of the epistemological analysis is so openly taken in account by logical analysis. (…). At the same time it appears as an effective epistemology in the measure that the reflection is oriented and invested in action.**

(Sinaceur, H. 1991, Logique : mathématique ordinaire ou épistémologie effective ?, in *Hommage à Jean Toussaint Desanti*, TER)

* *Our translation*

Syntax / semantics / Pragmatics

We assume a logical perspective on semantics, that suits with the following definitions referring to Morris (1938) or Eco (1971)

Semantics concerns relation between signs and objects they refer to.

Syntax concerns the rules of integration of the signs in a given system

Pragmatics concerns the relationship between subjects and signs : signs perceived according to their origin, the effects they produce, and the way they are used.

According with Da Costa (1997), it is necessary to take in account these three aspects for a right understanding of logical mathematical field.

A example pertaining to interdisciplinarity - Negation

Semantics : the negation of a proposition exchanges the truth value; the negation of a property or of relations exchange those objects which own the property/ the relation.

3 is an even number / 3 is not an even number

To be an even number / not to be an even number

a whale is a mammal / a whale is not a mammal

To be a mammal / not to be a mammal

Bologna is at the same latitude as Montpellier/Bologne is not at the same latitude as Montpellier

*To be at t*he same latitude as / *not to be at the same latitude as*

.

A example pertaining to interdisciplinarity Negation

Syntax : the negation of a given proposition follows precise rules with respect of the concerned language.

In French : for singular proposition, and for universal propositions, we apply “ne ..pas” on the verb :

Tous les animaux marins sont des mammifères/ Tous les animaux marins ne sont pas des mammifères

All marine animals are mammals / Not all marine animals are mammals.

Predicate calculus : For all x, P(x) / Exist x, non P(x)

As a matter of fact, the syntax of negation in French is not congruent with the syntax of negation in Predicate calculus, while Arabic for example is.

A example pertaining to interdisciplinarity - Negation

Syntax : the negation of a given proposition follows precise rules with respect of the concerned language.

In French, for existential propositions, we use the quantifier “aucun” (*no*).

Certains diviseurs de 12 sont pairs / Aucun diviseur de 12 n’est pair

Some divisors of 12 are even / no diviseurs of 12 are even

Certains animaux marins sont des mammifères / Aucun mammifère marin n’est un mammifère

some marine animals are mammals/ no marine animal are mammals

A example pertaining to interdisciplinarity-Negation

Pragmatics : Some linguistic forms may lead to referential ambiguities

"Tous les nombres ne sont pas pairs" (1) (*not all numbers are even*) is sometimes interpreted as "Aucun nombre n'est pair" (2) (*No prime number is even),* the contrary in Aristotle' sense.

This interpretation is reinforced by the possibility of changing " ne sont pas pairs" (*are not even*) in "sont impairs" (*are odd*) in sentence (1), that gives "Tous les nombres sont impairs", synonym of (2)

A example pertaining to interdisciplinarity - Negation

Pragmatics : Some linguistic forms may lead to referential ambiguities

Tous les animaux marins sont des vertébrés / tous les animaux marins ne sont pas des vertébrés/ tous les animaux marins sont des invertébrés.
All marine animals are vertebrates / not all marine animals are vertebrates / all marine animal are invertebrates
Tutti gli animali marini sono vertebrati / Tutti gli animali marini non sono vertebrati / Tutti gli animali marini sono invertebrati

A example pertaining to interdisciplinarity - Negation

Generally, but not always, the context permits to choose the right interpretation.

When translating from a langue to another, the interpretation might change

This leads to actual difficulties for students in practising mathematics, in particular, confusion between *negation* and *contrary*, that are largely underestimated in the teaching of mathematics, whatever the level. I suspect this is also the case in other disciplines.

The logical formalisation requires to choose an interpretation.

Example of activity around Negation

Asking to provide negation of significant sentences in the different disciplines and differing languages.

Collecting the answers – one often get a great variety

Discussing which among the provided sentences fulfil the semantic criterion

Asking to formalise the sentences in predicate calculus.

Moving back to interpretation/discussing on negation versus contrary

In cases of sentences with a complex logical structure, this might need to first provide a paraphrase as a first step before formalising.

Developing such activities relying on the use of logical analysis of language as a tool allows:

1/to make visible for teachers and students that there are unavoidable ambiguities and implicit in discourse in class;

2/to anticipate possible difficulties that students could face for interpreting the statements at stake in their classroom activities;

3/ to open possibilities for interpreting students' answers and arguments, even in cases where there could appear as incoherent at a first glance.

The logical analyse of language is a tool to reveal the complexity of statements and to anticipate the effects on the tasks proposed to pupils and students on the one hand (a priori analysis); for analysis the work of students on the other hand (a posteriori analysis)..

> "The maxim of translation underlying all this is that assertions startingly false on the face of them are likely to turn on hidden differences of languages.[…]. The common sense behind the maxim is that one's interlocutor silliness, beyond a certain point, is less likely than bad translation – or in a domestic case linguistic divergence." (Quine, 1960, p. 59)

Quine, W.V.O.: Word and Object. The M. I. T. Press, Cambridge, Mass. (1960),

Relations versus property - Symmetry
(From a work with Aurélie Chesnais)

An everyday notion, familiar to students;

Taught from primary school to university;

A central notion among the geometrical transformations;

A central notion in the curriculum in France;

Involved in several disciplines;

Involved in many professional activities;

It is emblematic of the gap between pragmatic validation and theoretical proofs in Geometry.

It involves a dialectic between the operational form of knowledge and the predicative form of knowledge (Vergnaud 2002)

- *Symmetry* as a property of a given figure.
- *Axial symmetry* as a tertiary relationship involving two figures and an axis – a space transformation
- *Axial symmetry* a geometrical transformation involving points – a plane transformation
- *Axial symmetry is an isommetry*
- *Symmetry in mathematics beyond geometry (e.g. Algebra)*
- *Symmetry* as a type of invariance: the property that something does not change under a set of transformations. – Maths, Physics, Chemistry, Arts.

We need to ground mathematical proofs also on geometric judgments which are no less solid than logical ones: *"symmetry", for example, is at least as fundamental as the logical "modus ponens"*; it features heavily in mathematical constructions and proofs. (Longo, 2012)

Longo, G. (2012) Theorems as constructive visions. In G. Hanna & M. de Villiers (eds) *Proof and proving in mathematics education. The 19th ICMI Study*. Springer

A lot of work has been done in research in Didactic of mathematics mathematics on Axial symmetry.

The relevance of the logical analysis has been addressed in a workshop at the Summer School of the ARDM in 2017, (Chesnais, Durand-Guerrier, Perrin 2019) considering the dialectics between operational form of knowledge and predicative form of knowledge pointed out by Vergnaud.

This could be a stating point for an interdisciplinarity module on Symmetry ?

Chesnais, A., Durand-Guerrier, V., Perrin M.J. (2019) regards croisés sur quelques enjeux didactiques de l'enseignement à la transition école collège en France. Actes de la XiXème école d'été de l'ARDM, Paris, Août 2017.

Complexity comes not only from doing, but also from putting something into words and saying it. Enunciation plays an essential part in the conceptualization process. One of the difficulties that students encounters when they learn mathematics is that some mathematical sentences and symbolic expression are as complex as the situations and thought operations needed to thought with them. Some researchers even consider that the difficulty of mathematics is mainly a linguistic difficulty. This view is wrong, because mathematics is not a language, but knowledge. Still, understanding and wording mathematical sentences plays a significant role in the difficulties students encounter. (Vergnaud, 2009, pp.89-90)

Vergnaud, G. (2009) The theory of conceptual fields. *Human Development, 52*. 83-94.

Truth versus validty

Aristotle syllogistic as a tool to make clear this distinction

Tarski's Methodology of deductive sciences to clarify the relationship between logical validaty and truth in an interpretation.

Geometry as modeling of the sensitive world

Research problem relying on Platonic and archimedian solids (Dias & Durand-Guerrier) and tiling of the plane by regular polygons (M. Front)

Interface mathematics/computer sciences (*Simon****).***

Thank you for your attention

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Identities - Kick-off meeting

Montpellier approach to interdisciplinarity and examples of modules and activities that can be carried out in the Summer schools: the epistemological lever for curricular mathematics-physics interdisciplinarity

Thomas Hausberger & Nathan Lombard

Bologna
November 19th, 2019

Outline of the presentation

The Mathematics, Physics & Philosophy IREM group

- IREM: Instituts de Recherches sur l'Enseignement des Mathématiques
`http://www.univ-irem.fr/`
- IREM institutes are:
  - *research institutes* focusing on the specific perspectives and problems that are emerging at all levels in mathematics education today;
  - *teacher training institutes* through actions strongly based on fundamental and applied research;
  - institutions for the *production and distribution of educational materials* (articles, brochures, textbooks, journals, software, multimedia documents, etc.).
- Composition of the Mathematics, Physics & Philosophy group of the IREM Montpellier: Thomas Hausberger (didactics and epistemology of mathematics, coordinator), Manuel Bächtold (didactics and epistemology of physics), Dominique Guin (didactics of mathematics), Daniel Guin (mathematician), Henri Reboul (physicist), Véronique Pinet & Thomas François (philosophy high school teachers), Sandra Bella, Christophe Pointier & Patrice Marie-Jeanne (mathematics high school teachers)

Missions of the MPP IREM group and design principles

- Promote critical and reflexive thinking on sciences based on the work of *epistemologists*
- Promote synergy between *mathematics / physics and philosophy* teachers, especially in the final year of upper high school;
- Reconcile sciences and humanities by emphasising that scientific activity is a human endeavour
- Put into perspective the specificities of mathematical thinking (language, approach, nature of objects...)
- Select some topics from the *official syllabus* where *interdisciplinary work in science & philosophy* is appropriate
- Develop pedagogical and didactical scenarios, experiment and analyse their effects. These scenarios may involve academics group members who embody the world of research. They will be recorded in well-documented resources.

The MPP complex numbers resource kit

- A mathematical activity: *the cubic equation, a complex story*
  `https://hal.archives-ouvertes.fr/hal-02319858`
  ☞ A teaching and learning approach inspired by history, introduction of complex numbers following Cardan and Bombelli
- A physics-philosophy activity: *the frustrated total internal reflection*
  `https://hal.archives-ouvertes.fr/hal-02319884`
  ☞ A concrete yet somewhat surprising application of complex numbers to physics and a philosophical discussion of the reasons for the applicability of mathematics to phenomena
- Activities for *further discussion in the philosophy classroom* of the previous and supplementary material ("*reason and reality*" is a theme of the philosophy syllabus of the final year of upper high school)
  `https://hal.archives-ouvertes.fr/hal-02319748` and
  `https://hal.archives-ouvertes.fr/hal-02319833`

Educational goals

- contribute to *give meaning* to complex numbers by means of an *epistemological* discussion of these *abstract objects* in the *philosophy* course
- show students a *concrete application* of complex numbers in *physics*
- give students a glimpse of how these mathematical objects can be used in their possible *future university education* in physics
- the three previous points can *motivate* students to work on complex numbers in mathematics courses
- the *co-intervention* of teachers in sciences and philosophy to increase the classroom dynamic and students' interest and motivation
- material to cover the part of the *philosophy syllabus* dedicated to epistemology with concrete, precise and relevant references to actual scientific practices

Teacher training (in-service teachers) on the complex numbers kit

- A *2h workshop* during a conference on "the 2012 reform of the grade 12 syllabus - its impact on the secondary-tertiary transition"
- Audience: *math teachers and IREM teacher trainers*
- The resource kit was used to discuss complex numbers as:
  - a mathematical object situated at the secondary-tertiary transition in the teaching of mathematics
  - an example to illuminate the links between mathematics and physics (epistemology)
- An article published in the proceedings: `https://hal.archives-ouvertes.fr/hal-00879219`

Didactic principles for the design of the activity

- An epistemological obstacle: the *epistemic irrational*

The epistemic irrational appears in the very process of knowledge, when this process unexpectedly meets a property of its object that prohibits the continuation of this process as it stands. In this field, it is therefore a typical case of irrationality as an obstacle, considered in the act of knowing. But it may also happen that the subject deliberately uses such a contradiction which he himself introduces into the object and assumes, at least temporarily, to obtain new results. It is always a question of a moment of the work of constitution of the scientific object, and the irrational in no way qualifies an attitude and practical behaviour of the actor. (G. Gaston-Granger, the irrational, Odile Jacob, 1998 chap. II)

Morellet, arcs complémentaires

Didactic principles for the design of the activity

- An approach *inspired by history*.
- Do not start from the quadratic equation but the cubic, which leads to a *cognitive conflict*: the Cardan formula cannot be applied any more whereas there are three roots in the real numbers.
- Give yourself time to present the process of elaboration of the new tools (complex numbers) and discuss the relevance of notations.
- Thus *restore the rationality* of complex numbers: the introduction of the square root of a negative number, in the historical context of Cardan's formula, appears as a deliberate process justified by its success in resolving the cognitive conflict a posteriori.

Extract from the tutorial sheet

« J'ai trouvé une autre sorte de racine cubique d'expressions (...) très différente des autres. Celle-ci est issue du Cas sur le Cube du Tant[1][l'inconnue] *égal au Tant et à une Constante , quand le cube du tiers du nombre de Tants est plus grand que le carré de la moitié de la Constante, comme cela va être démontré pour ce Cas.*

c est le nombre de « tants » et d est la constante

$x^3 = cx + d$

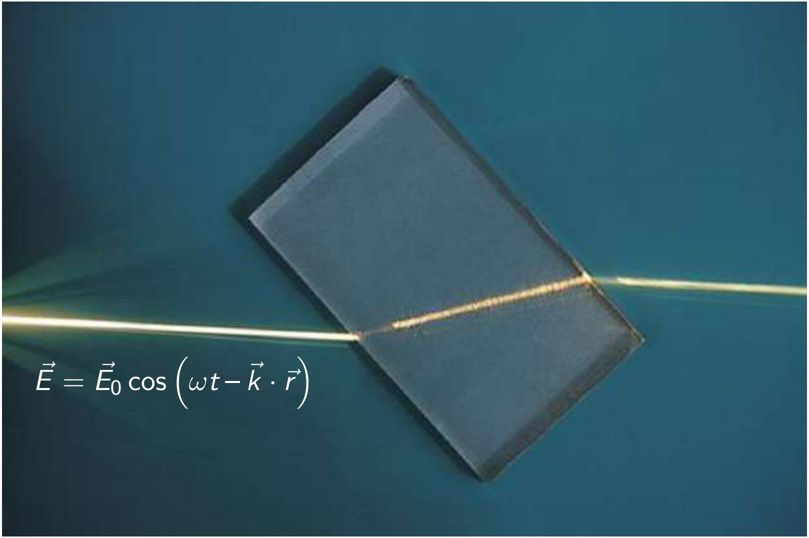
$(\frac{c}{3})^3 > (\frac{d}{2})^2$

Cette sorte de racine carrée contient dans son algorithme de calcul, des opérations ***différentes*** *des autres et a un nom différent, parce que(...), la racine carrée de leur différence ne peut s'appeler ni plus ni moins c'est pourquoi je l'appellerai plus que moins [pdm Rq] quand celle-ci devra être ajoutée et, quand elle devra être ôtée, je l'appellerai moins que moins [mdm Rq](...)*

Dans le cas de l'équation $x^3 = 15x + 20$, on a $(\frac{d}{2})^2 - (\frac{c}{3})^3 = -121 < 0$. Cet exemple correspond au cas décrit par Bombelli.

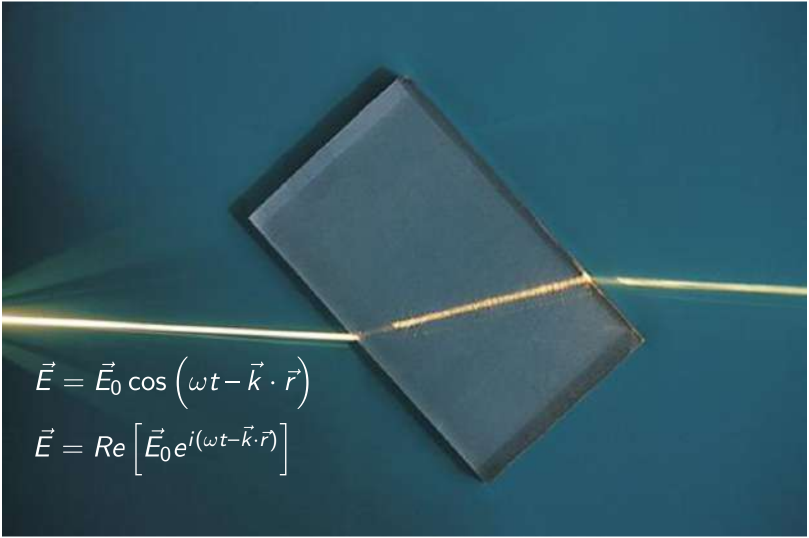
Bombelli décide de faire comme si - 121 était le carré d'un nombre imaginaire et le note pdm Rq 121. Les mathématiciens du XVII^ème^ et XVIII^ème^ siècle le noteront en utilisant le signe $\sqrt{-\ldots}$, ce qui donne $\sqrt{-121}$

Total reflection: light waves as complex exponentials

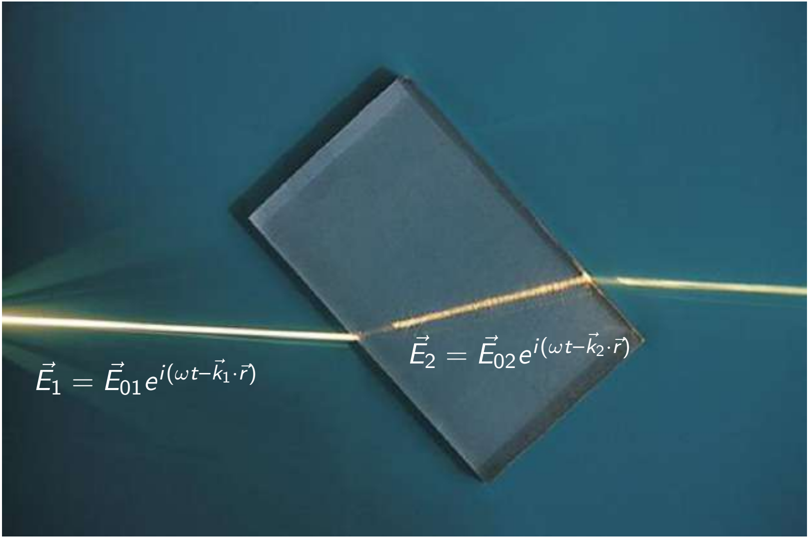


$$\vec{E} = \vec{E}_0 \cos\left(\omega t - \vec{k} \cdot \vec{r}\right)$$

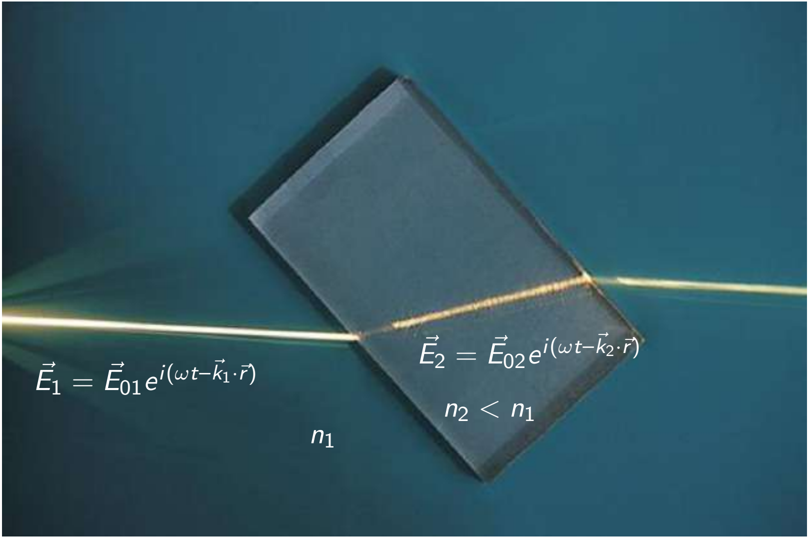
Total reflection: light waves as complex exponentials



Total reflection: light waves as complex exponentials



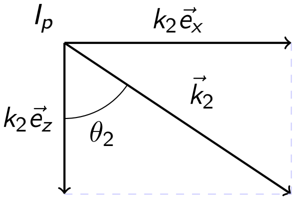
Total reflection: light waves as complex exponentials



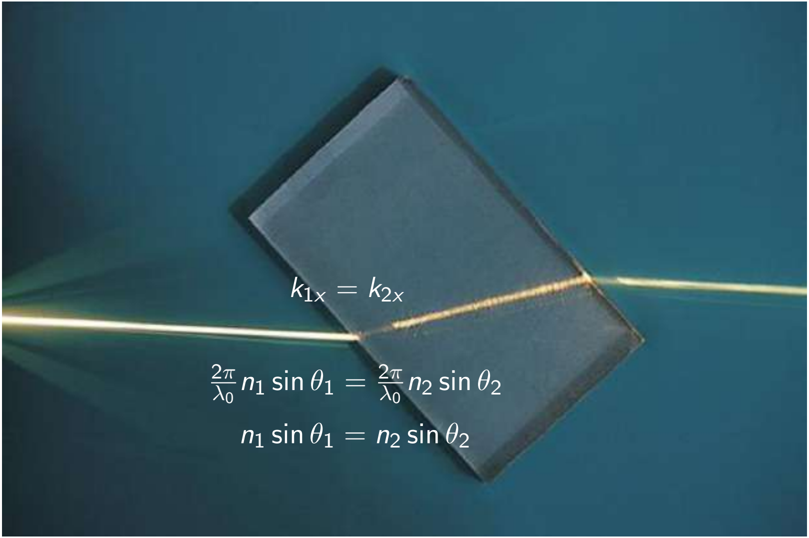
The emerging wave vector

Let:

$$\vec{k}_2 = k_2\left(\cos\theta_2 \vec{e}_z + \sin\theta_2 \vec{e}_x\right) \underset{k_2=\frac{2\pi}{\lambda_0}n_2}{\Longrightarrow} \vec{k}_2\cdot\vec{e}_x \equiv k_{2x} = \frac{2\pi}{\lambda_0}n_2\sin\theta_2$$



Snell's law from above



$$k_{1x} = k_{2x}$$

$$\frac{2\pi}{\lambda_0} n_1 \sin\theta_1 = \frac{2\pi}{\lambda_0} n_2 \sin\theta_2$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

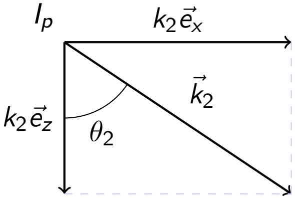
The evanescent wave

Let:

$$\vec{k}_2 = k_2\left(\cos\theta_2 \vec{e}_z + \sin\theta_2 \vec{e}_x\right) \underset{k_2=\frac{2\pi}{\lambda_0}n_2}{\Rightarrow} \vec{k}_2\cdot\vec{e}_x \equiv k_{2x} = \frac{2\pi}{\lambda_0} n_2 \sin\theta_2$$

$$k_2^2 = k_{2x}^2 + k_{2z}^2 = (k_1 \sin\theta_1)^2 + k_{2z}^2$$

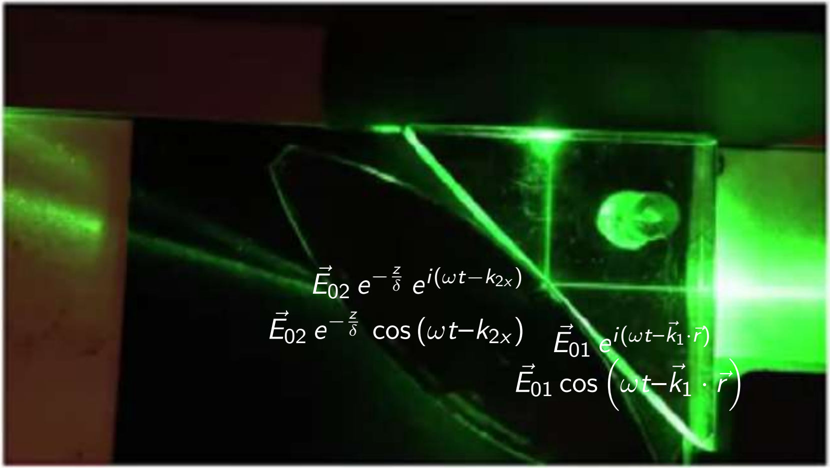
$$\Rightarrow k_{2z} = \sqrt{k_2^2 - (k_1\sin\theta_1)^2}$$



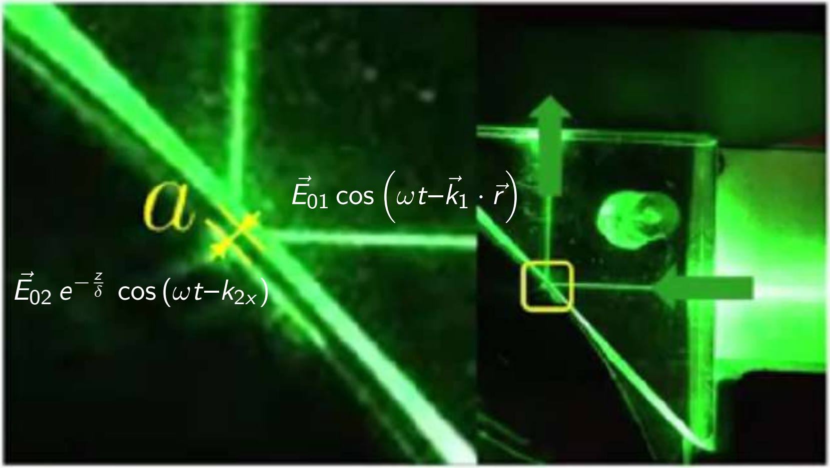
$$\theta_1 > \theta_{\lim} \qquad k_{2z} = \pm i\sqrt{(k_1\sin\theta_1)^2 - k_2^2}$$

$$\boxed{\vec{E}_2 = \vec{E}_{02}\, e^{i(\omega t - k_1 \sin\theta_1 x)}\, e^{-\sqrt{}\, z}}$$

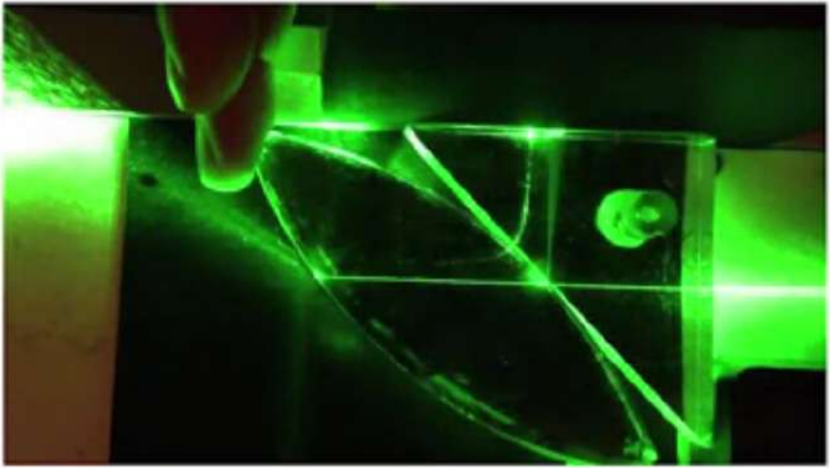
A picture of the evanescent wave



A picture of the evanescent wave



An optical tunnel effect



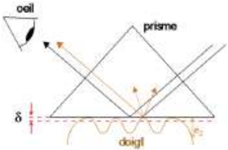
Two approaches for students

◇ Experiment first approach

- Finger print experiment
- Limits of the geometrical model
- Wave optics view on the total reflection
- Introduction of the complex degree of freedom

◇ Math first approach

- Mathematical recovery of Snell's law
- In what condition is the emitted wave vector complex
- Study of the evanescent wave function and its parameter δ
- Can it be an interpretation of a physical phenomenon?



Epistemological stakes for mathematics: complex numbers and mathematical formalism

- The study of complex numbers sheds light on the construction, development and use of *mathematical formalism*:
  - complex numbers are *formal* objects: they are described by new symbols (i) or defined as pairs called numbers only because they are endowed with operations;
  - the construction of this formalism has a *history* that allows its *problematisation*;
  - the construction principle was established as a method by Hilbert: adding ideal elements, for convenience and simplicity, in order to obtain the permanence of certain laws;
  - this formalism shows its fruitfulness through unexpected applications (geometric interpretation, physics), which justifies the *gain of abstraction a posteriori*
- Didactic consequence: an epistemology-rich activity on complex numbers may thus facilitate the *appropriation by students of formal abstract approaches in mathematics*.

Epistemological stakes for physics: complex numbers and reality

- Questions raised by this activity
  - *Validity of a law* in physics : existence of solutions of equations in mathematics, interpretation of their existence or non-existence
  - Concept of *problematical facts*, and how experiments can help discovering one ; why it is good news
  - Meanings of reality: mathematical vs ontological
- Epistemological phenomena touched upon by this activity
  - *Effectiveness of mathematics* in physics
  - Tool metaphore to approach the math-physics relationship: math as a *microscope* (concepts described in details) or a *winch* (aspects of reality dredged up on the surface) ; math equips the predictive spirit of physics
  - Difference of *nature* between mathematical objects, physical concepts and phenomena
  - Some answers to these questions: Aristotle, Galileo, Hume, Lambert, Duheim...

Further epistemological stakes (not covered): interrelations of algebra and geometry, opening up to physics

- The discovery of complex numbers will allow a significant extension of the possibilities of *algebraisation of geometry* and *geometrisation of algebra* in further historical developments ($\mathbb{H}$, $\mathbb{O}$, Clifford algebras)
- Geometric algebras: *language of 20th century physics* (Dirac's relativistic electron)
- This path of thought leads to the major *unification programmes* (Langlands programme in mathematics, super-chords in physics)
- Philosophy: algebra creates numbers to translate geometric properties and conversely, geometry provides intuitions to describe the properties of equations of curves, surfaces, algebraic varieties,...
- Origin: Descartes' rewriting of geometry in terms of algebra, an imperfect match. The breakthrough of Argand allowed to interpret operations on complex numbers geometrically, with the birth of a new frame: vectors.

The MPP non-euclidean geometries resource kit

- An integrated math-physics-philosophy resource: "*Geometry and reality: a questioning about truth* https://hal.archives-ouvertes.fr/hal-01469911
    - A mathematical activity "*towards a new geometry*" on Euclide's axioms and the geometry of Poincare's upper half plane, which is non-euclidean
    - A brief presentation of the GPS as an application of non-euclidean geometry to physics through general relativity theory
    - A philosophical problematisation of truth for the philosophy course
- *Satellite documents* for the self-training of in-service teachers
    - history of non-euclidean geometries (nEG) https://hal.archives-ouvertes.fr/hal-01442924
    - mathematical aspects of nEG https://hal.archives-ouvertes.fr/hal-01442936
    - epistemological stakes of nEG https://hal.archives-ouvertes.fr/hal-01442915
    - general relativity theory and cosmology https://hal.archives-ouvertes.fr/hal-01454764
- A research article that accounts for the design, implementation and results: https://hal.archives-ouvertes.fr/hal-01668202

Educational goals of the interdisciplinary math-(physics)-philosophy scenario

Extract from the resource identification sheet:

> *First of all, it is a questioning on truth, in the sense of the adequacy between theoretical discourse and reality, that is implemented in the proposed scenario.*
> *- In mathematics, students will be shown that there are several geometries, which differ in both their axioms and theorems, but which are each logically coherent and apply to reality.*
> *- In philosophy, we will question, with the students, the implications of this theoretical pluralism. Its acceptance implies a change of perspective on the nature of mathematics and more generally the relationship between reason and reality.*

Teacher training on the nEG resource kit

- Audience:*math teachers and philosophy teachers*
- A *hybrid 2 days* face-to-face + satellite documents available on the online platform
- Day 1: *mathematical and epistemological background* on nEG
  - A conference on the history of nEG
  - A mathematical presentation of nEG in disciplinary groups
  - An interdisciplinary panel on the epistemological stakes of nEG
  - A workshop dedicated to the study of a philosophical text by Poincare
- Day 2: *educational aspects around the interdisciplinary activity*
  - A workshop dedicated to the mathematical activity, in the pupil's posture
  - A workshop for math teachers on didactical aspects of the activity
  - A workshop for philosophy teachers on ways to implement a philosophical discussion based on this material
  - A workshop dedicated to the presentation of educational effects of the scenario with videos and research results (survey to assess students' learning)

Extract from the tutorial sheet

I. Back to Euclid (around - 300 BC)

Postulates (or axioms) are what the mathematician considers to be true and serve as the basis for theory. Once these axioms have been established, all properties must be demonstrated in a deductive manner. When Euclid wrote the foundations of geometry (plane) in his treatise, The Elements, he made the following five assumptions: [...]

The question that mathematicians ask themselves is whether the fifth assumption cannot be deduced from the others. In other words, is it necessary to admit the fifth premise or can it be demonstrated by using the other four?

It is likely that Euclid himself was asking himself the question. The reasons for this are as much the formulation of the statement of the historical postulate as the fact that Euclid establishes the first 28 proposals of its treatise without resorting to this fifth postulate.

In an attempt to demonstrate that the fifth postulate can be deduced from the other four, an absurd reasoning is made: [...]

In fact, we can construct a different geometry from Euclid's which satisfies the first four postulates and the following postulate:

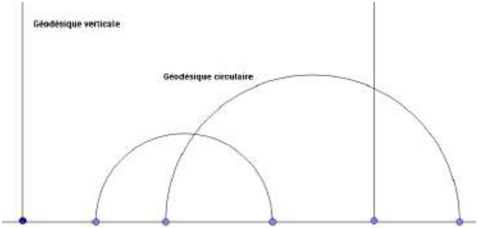
Postulate 5' : for any straight line D and any point A outside D, there are an infinite number of lines parallel to D passing through A.

Extract from the tutorial sheet (II)

II. A new geometry: Poincaré's half-plane H (proposed in 1882)

In the plane endowed with an orthonormal basis, let us consider the open half-plane P defined by $y > 0$. We note λ the straight line $y = 0$ which borders P. The space H is the set P with a geometry in which the "lines" - or rather what is the analogous of the lines in euclidean geometry, namely the curves that achieve the minimum distance between two points, called geodesics - are:

- type 1: the Euclidean half straight lines (origins excluded) parallel to the y-axis (we will say in the following "vertical geodesics").
- type 2: Euclidean semicircles (excluding ends) with centres at λ (hereinafter referred to as "circular geodesics").



The shapes of these geodesics come from the fact that, to calculate the lengths, H is provided with a distance different from the Euclidean distance (the one measured with a ruler). Specifically, [...]

Extract from the tutorial sheet (III)

A. The geodesics of H and the five postulates of Euclid
[...] We have therefore just verified that the geometry of H satisfies the equivalent of Euclid's first four postulates as well as the equivalent of postulate 5' (thus implying the negation of the equivalent of postulate 5). The fifth postulate is therefore not the logical consequence of the first four.

B. Some properties of this geometry

1. Parallelism of geodesics
In this new geometry, is the euclidean theorem "if two lines are parallel to a third, they are parallel to each other" still valid by replacing the term "line" by "geodesic"?

2. Triangles
a) Draw "triangles" in H (there are several possible shapes). Calculate an approximation of the sum of the angles, noted S, for each of these triangles.
b) What would S be worth if it were possible to place the 3 vertices of the triangle on λ? Deduce that there are triangles such that S is as small as one wants.

3. Rectangles
We will admit that in this new geometry the sum of the angles of a "triangle" is lower than π and that the area of a "triangle" (with angles α, β, γ) is equal to $\pi - (\alpha + \beta + \gamma)$. A "rectangle" is a quadrilateral with four right angles. Show that, in this new geometry, there can be no true "rectangle" (i.e. not flattened).

nEG and spacetime theories

Newtonian gravity

It can be formulated as a consequence of a curved geometry, even though it remains inconsistent with Special Relativity. This theory then exhibits a gravitationnal timeshift.

Special relativity (SR)

Principle of relativity
c-invariance in vacuum
} $\Rightarrow$ the geometry of a (ct, x) slice of flat spacetime is not euclidian

So (flat) nEG is an immediate consequence of the theory's fundamental principles

First observation

nEG $\not\Leftrightarrow$ new physics principles

nEG and spacetime theories (II)

Underdertemination in general relativity

Equivalence Principle $\Rightarrow$ gravitational timeshift $\Leftarrow$ spacetime is curved
or gravity affects clocks ?

General relativity (GR)

energy curves spacetime
"gravity is geometry"
$\Downarrow$
Equivalence Principle $\Rightarrow$ gravitational timeshift $\Leftarrow$ spacetime is curved

Special relativity then constraints the spacetime metric and its dynamics.

Main observation

General relativity shows powerful physics principles can leverage nEG to achieve greater predictive power through more general and precise laws

Physical consequences of General Relativity

A one-century-long harvest

- The more profound: the birth of physical cosmology
- The prediction of a new astronomical era, that of gravitationnal waves observations (confirmed in 2016)
- The description of new astrophysical objects, such as black holes
- Numerous technical applications, such as the GPS

Relativity theories and the GPS

- SR: Relativity of simultaneity
- SR: Time dilatation : $\sim 7\ \mu s.d^{-1}$
- GR: Gravitational timeshift : $\sim 45\ \mu s.d^{-1}$
- GR: Curved path of electromagnetic waves

Error on the position otherwise: $\sim 11\ km.d^{-1}$

Epistemological stakes for mathematics

- The *foundations of mathematics*
  - an opportunity to present to students and question *Euclid's axiomatics* of geometry;
  - local axiomatics used in high school are not enough to bring out the notion of axiom;
  - philosophy raises the question of premises and the *distinction between axiom and theorem*.
- The *leap towards abstraction* that is achieved with nGE
  - overcoming an *obstacle* that is both *epistemological* (euclidean geometry as the only possible geometry) and *ontological* (the fact that the axioms of geometry must model the sensitive space);
  - an illustration of the *concrete-abstract dialectic*: mathematicians define axiomatic systems independently of, or even against, reality, but make representations of it (models);
  - more or less unexpected uses of mathematical abstractions may emerge (*unreasonable effectiveness* of mathematics): Einstein's general relativity;
  - epistemological questioning of nEG sheds light on the *meaning and scope of contemporary mathematical approaches* and thus prepares students for the *transition to university mathematics*.

Epistemological stakes for physics

- A change of *epistemological status* for spacetime...
  - Newtonian spacetime is a *setting* rather than a physical object; relativity changes this in two steps
  - SR relates kinematics and the geometry of spacetime
    GR relates dynamics and the geometry of spacetime
    $\rightarrow$ students can learn physics is not just about describing new objects, the way it *models* them matters
- ...which has profound consequences
  - The study of the newly-thought physical object "Universe" is historically *the first take of physics at cosmology*, which used to be the preserve of religion or philosophy
    $\rightarrow$ their methods may be compared by students
  - The FLRW-metric as the most general solution to Einstein equations satisfying the *cosmological principle* is an example of this new ability of physics to formulate philosophical propositions
    $\rightarrow$ students can reflect on the link between a phenomena and *a* solution of an equation
  - The object "Universe" was given a *dynamics* which suggested it had a *history*, and even a beginning to it (the Big Bang)
    $\rightarrow$ why and how interpreting an equation (case of $a(t)$ in FLRW)

Science-Philosophy interdisciplinarity

- nEG, an *interdisciplinary object*
  - an object defined and manipulated within mathematics;
  - but also according to a historical perspective (a renewed questioning of euclidean geometry in the 19th century);
  - or a philosophical perspective (relationships between theory and reality, notion of truth);
  - or according to the perspective of physics (non-euclidean geometrisation of space-time)
- How do *science and philosophy cooperate* in the form of complementary and interweaving approaches?
  - epistemological reflection as a way for students to give meaning to this specific mathematical object and through it to mathematical objects in general; meta-mathematical stakes described previously;
  - conversely, deepening the study of nEG rather than superficially mentioning this example, enriches philosophical reflection with students.

Proposed design principles

- Activities are designed on *curricular-related interdisciplinary topics* at the *secondary-tertiary transition* for the *math and physics* classrooms, in order to provide *educational material for teachers* (mainly high school teachers, but why not extend to the beginning of the bachelor?)
- *Epistemology* is used as a *lever* to discuss *epistemological stakes for math, physics and their interrelationships*. Further didactic design is required to transform the "socratic pedagogy" of philosophical inputs from IREM MPP activities into a more constructivist approach that may be carried out by science teachers.
- It is assumed that this will *help students make sense* of both mathematical objects and theories in physics thanks to the combined and intertwined disciplinary perspectives.
- These activities serve as the basis for a *higher-level design* in the form of *teacher training activities* which may combine online resources such as the IREM satellite documents and workshops in which chosen extracts from those activities are experienced by pre-service teachers in the pupil's posture. Further documents (epistemological sources, videos, extracts of students' work,...) are then provided for epistemological and didactical analyses to be conducted and discussed among teacher students.

Opportunities for several interdisciplinary modules

- A *module on complex numbers*
  - The math activity on Cardan and Bombelli
  - Pursuing the design of the FTIR as an interdisciplinary math-physics activity, letting students work out either the experiment-first approach or the math-first approach. Epistemological considerations can then bridge the two approaches to shed light on the relationship between math and physics.
- A *module on non euclidean geometries*
  - The math activity and potential further developments, e.g. "defining" activities (which definition for an hyperbolic rhombus? Equivalent properties from euclidean geometry do not hold true altogether)
  - Physics: derive from first principles the relativity of time and thus the need for a new geometry ; working out an approximate formula for the gravitationnal timeshift given the equivalence principle

Opportunities for several interdisciplinary modules

◇ A *module on quaternions*?

- Math: numbers to express algebraically the *composition of rotations* in space (generalising complex numbers)
- Although originally purely mathematical abstract objects, the quaternions nevertheless have an interpretation in physics, in the context of *relativistic quantum mechanics*.
  $\rightarrow$ For students it is a first example of an *abstract physical concept* (spin), experimentally detectable but describable only thanks to mathematics
- Epistemology of math-physics interdisciplinarity:
  Whereas complex numbers already are an epistemic irrational in the context of physics, quaternions show one can follow the thread even further.
  $\rightarrow$ Students may thus be introduced to the impact of *abstract algebraic structures in physics*, as en emphasis of their role in modern theoretical physics