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Highlighting Interdisciplinarity between Physics and Mathematics in Historical Papers on Special Relativity: Design of Blended Activities for Pre-Service Teacher Education

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A mio padre
Con mio padre

Per mio padre

Abstract

This thesis is framed within the Erasmus+ project titled IDENTITIES, whose aim is to develop interdisciplinary teaching materials for preservice teacher education. In collaboration with the research group in STEM education of Crete, a blended module on special relativity has been developed. The module is based on an analysis of the original texts by Lorentz, Poincaré, Einstein and Minkowski (written between 1904 and 1908), aimed to recognise the interplay between mathematics and physics implemented in the four papers. The analysis has been carried out by applying the 'Boundary Crossing and Boundary Object' research framework developed in 2011 by Akkerman and Bakker. The results of the analysis show that Lorentz Transformations can be read as a Boundary Object and this lens allows for different nuances of the interplay between mathematics and physics to be recognised in the four papers. A series of activities to be conducted in blended mode in a pre-service teacher education course have been designed with the goal of exploiting special relativity as a context to develop interdisciplinary skills.

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List of Abbreviations

ACE	Advanced Computing Environment
ICT	Information and Communication Technologies
IDENTITIES	Integrate Disciplines to Elaborate Novel Teaching approaches to InTerdisciplinarity and Innovate pre-service teacher Education for STEM challenges
LT	Lorentz Transformations
MER	Mathematics Education Research
MIX	Mixed Instructional eXperience
NRC	National Research Council
PER	Physics Education Research
STEM	Science Technology Engineering Mathematics
STR	Special Theory of Relativity

Introduction

This thesis is part of the European Erasmus+ project IDENTITIES, *Integrate Disciplines to Elaborate Novel Teaching approaches to InTerdisciplinarity and Innovate pre-service teacher Education for STEM challenges*, coordinated by the research group in physics didactics of the University of Bologna together with other four European universities. The thesis was carried out in collaboration with the research group in STEM education at the University of Crete thanks to the Erasmus+ Traineeship project from March to June 2021 under the supervision of Professor Dimitris Stavrou and Professor Costas Tzanakis.

The aim of this thesis is to develop blended activities for a pre-service teacher education. The activities are based on an interdisciplinary analysis of foundational articles for the Theory of Special Relativity, i.e. the papers from Hendrik Antoon Lorentz (1904), Henri Poincaré (1905, 1906), Albert Einstein (1905) and Hermann Minkowski (1908). The interdisciplinary analysis was carried out using the framework of "Boundary Crossing and Boundary Object" introduced by Akkerman and Bakker (2011).

The thesis is structured in five chapters.

In the first chapter there is an introduction about the concept of disciplines and interdisciplinarity aimed at introducing what is the state of the art concerning interdisciplinarity today in the STEM (Science, Technology, Engineering and Mathematics) education research field and why interdisciplinarity is so important. In this chapter there is also a description of the context in which this thesis takes place, namely the IDENTITIES project.

In the second chapter a structured literature review has been realized regarding the teaching and learning of the Special Theory of Relativity (STR) in Physics Education Research (PER) and Mathematics Education Research (MER) fields. This review is aimed at finding what are the main research strands in PER and MER regarding STR, which are the main difficulties that students face when approaching STR and which are the less studied strands.

In the third chapter there is the interdisciplinary analysis of the original papers. This chapter begins with a description of the historic moment of late 1800' when the Newtonian System "started to fall" as an absolute system, while the Euclidean Geometry "fall" was already occurring during the same century. This decline was marked by different reasons: the birth of theories like Statistical Mechanics and Non-Euclidean Geometries, the "revolution" of Felix Klein with his group theory and the experiments made to prove the existence of the ether. All of these circumstances contributed to the works of Lorentz, Einstein, Poincaré and Minkowski, that are carefully presented in the second section of the chapter. After this introduction, the third section is dedicated to the discussion of the interdisciplinary framework "Boundary Crossing and Boundary Object" built by Akkerman and Bakker. In the section there is a description of the possible mechanisms of learning that can be found in the processes of boundary crossing, that are Identification, Coordination, Reflection and Transformations. These interdisciplinary lenses therefore are used in the fourth chapter to understand what type of learning mechanisms and interdisciplinarity is present in STR original papers, i.e., Lorentz 1904 "Electromagnetic phenomena in a system moving with any velocity smaller than that of light", Poincaré 1905 and 1906 extended version of "On the Dynamics of the Electron", Einstein 1905 "On the electrodynamics of moving bodies", and Minkowski 1908 "Space and Time".

The fourth chapter is focused on the theme of blended learning and is functional to the activities that are presented in the following chapter. Blended learning is a mixed learning approach that developed in the last twenty years thanks to the development of Information and Communication Technologies (ICT) and consequently also of classical learning models. The chapter is divided in three main parts: the first part is focused on the definition of the Blended approach, with a focus on the main features of this kind of approach and the reasons why this approach developed so much in the last years; the second part is focused on the means and the methodologies with which is possible to implement a Blended approach; the third chapter is focused on the issues that this kind of approach has in schools and society.

The fifth chapter is dedicated to the design of four activities that will be implemented in a blended modality next year in an IDENTITIES module for pre-service teachers in formation. The activities are based on the analysis realized in the third chapter on the original STR papers and are focused on different aspects of the analysis.

Chapter 1 - Introduction to Interdisciplinarity

1.1 State of the art

Bridging the gap between science and society is crucial, if not mandatory, in this age of uncertainty and in what Rosa defined "society of acceleration" (Rosa, 2013). The great changes that have taken place in society in recent years, such as the advent of social networks and the evolution of new technologies, have enlarged the gap between the scholastic and academic world (schools, universities, and research) and everyday life. The disconnection and misalignment between what is taught and what students need to address daily challenges is increasing everyday. This disconnection depends on several factors. One of the most important is how school curricula are strictly structured and organized in disciplines, while modern society and research require a more open vision that is intra-multi-trans-disciplinary.

In the last twenty years, European science policy had the central goal to foster "Interdisciplinarity in Research and Innovation" (Müller & Kaltenbrunner, 2019). Both the previous European Union framework program 'Horizon 2020' (https://horizon2020.apre.it/) and the new 'Horizon Europe' (https://horizoneurope.apre.it/) stress the importance of interdisciplinarity and collaboration across borders and disciplines to "promote societal, ecological and economic transformations by involving, collaborating with - and building consensus among citizens and practitioners on research and innovation roadmaps and priorities" (European Commission, 2021, 4).

In the 2013 report by the Committee on Undergraduate Physics Education Research and Implementation from the National Research Council of the National Academies of United States it is clearly stated that education in physics from primary education up to undergraduate courses "forms the gateway into technological competence and expertise" in every important challenge in our world. (National Research Council [NRC], 2013, viii). Higher education in our era has the goal to prepare students for the international labour market, where the requested abilities, to be competitive, are "learning and understanding new things, understand complex systems, manage large sets of data, think creatively and critically, communicate and collaborate" (NRC, 2013, 10). Skills acquired through the study of Physics are of extraordinary importance for the challenges of our world, but also for STEM (Science, Technology, Engineering, and Mathematics) research: disciplinary practices of empirical and theoretical inquiry are necessary for all STEM disciplines. In these fields these developments are forcing us to change and rethink the role of teachers and researchers (NRC, 2013). Adopting an interdisciplinary approach is therefore paramount, because "an interdisciplinary learning approach integrates the disciplines and diffuses their limits, passing through different levels of cognitive ability in pursuit of developing a holistic thought process. In this manner, students can make meaningful connections that allow them to process knowledge to produce an interdisciplinary understanding that is applicable to reality" (Martín-Páez et al., 2018, 802). Interdisciplinarity is necessary to understand the challenges of today's society due to their complexity and their multifaceted nature. In doing so, it is fundamental to focus on the role that disciplinary epistemologies, methodologies and practises play in the shift from a disciplinary approach to an interdisciplinary one. As Honey et al. (2014) stated:

"Despite the arguments for making connections across the STEM disciplines and the increased number of efforts to design learning experiences that will foster such connections, there is little research on how best to do so or on whether more explicit connections or integration across the disciplines significantly improves student learning, retention, achievement, or other valued outcomes" (p. 22).

In order to understand how interdisciplinarity should and can be developed, it is important to focus on why and how curricula are structured in disciplines, from a historical, methodological, and epistemological point of view.

1.2 History and definition of disciplines

As stated by Krishan (2009), "the main problem with the notion of 'interdisciplinarity' seems to be that many people who use it do not make explicit what exactly they understand under a discipline or when exactly a disciplinary boundary is crossed with what kind of consequence" (p. 6). Thompson Klein (1990) pointed out Academic disciplines were invented in the Middle Ages, with the term 'discipline' originating from the Latin word *discipulus*, which means pupil, and *disciplina*, which means teaching. This process of "disciplining knowledge "was due to external demands, while the new specialization into disciplines of the 19th and 20th century was due to internal reasons, like social changes, economical necessities, and technological development. Every new discipline was established with the aim of dealing exclusively with a particular object or topic not covered by other disciplines.

But defining what is meant by the term "discipline" is not easy. Krishnan described six criteria and characteristics to identify if a subject can be considered a distinct discipline. These are:

1. disciplines have a particular object of research;
2. disciplines have a body of accumulated specialist knowledge referring to their object of research;
3. disciplines have theories and concepts that can organise the accumulated specialist knowledge;
4. disciplines use specific terminologies or a specifical technical language adjusted to their research object;
5. disciplines have developed specific research methods according to their specific research requirements;
6. disciplines must have institutional manifestation in the form of subjects taught at universities or colleges, academic departments and professional associations connected to it (Krishnan, 2009, p. 9).

Apart from these characteristics, disciplines are shaped in such a way that they include the knowledge that, along the various historical periods, are retained useful for addressing challenges requested by society. For this reason, disciplines change during time, they separate from each other and follow different paths, like psychology found its own way from medicine and philosophy. This continuous movement, together with the new demands of modernity, led during the last twenty years to the development of concepts such as interdisciplinarity, multidisciplinarity or transdisciplinarity, in that the new challenges that society faces are not addressable from an isolated point of view.

Considering the transition between disciplines, "disciplinary boundaries are often blurred and there is much research to show that disciplines are not static, instead knowledge, methods and theories are constantly moving across disciplinary boundaries and the disciplines are constantly shifting [...] Interdisciplinarity presupposes disciplinarity such that one is integral to the other" (Millar, 2020, 939). As stated by Lenoir & Hasni (2016), "there can be no interdisciplinarity without disciplinarity" (p. 2448).

Thompson Klein (2010) described the differences between interdisciplinarity, multidisciplinarity or transdisciplinarity:

- Multidisciplinarity is defined as "an approach that juxtaposes disciplines. Juxtaposition fosters wider knowledge, information, and methods. Yet, disciplines remain separate, disciplinary elements retain their original identity, and the existing structure of knowledge is not questioned" (p. 17). The main characteristics in this approach are juxtaposing, sequencing, and coordinating.
- Interdisciplinarity is defined as an approach in which integration and interaction between disciplines are proactive. There are several ways in which interdisciplinarity can manifest itself,

that are Narrow ID, Broad or Wide ID, Shared ID, Cooperative ID. The differences between these ways are due to the relation between the interacting disciplines, if they share methodologies, paradigms, or epistemologies (pp. 18-19). The main characteristics in this approach are integrating, interacting, linking, focusing, blending.

- Transdisciplinarity is defined as "a common system of axioms that transcends the narrow scope of disciplinary worldviews through an overarching synthesis" (p. 24). The main characteristics in this approach are transcending, transgressing, transforming.

1.3 The challenges of STEM Education

A paramount field in which interdisciplinarity is the key element is STEM. The basic idea behind the STEM movement started to be emphasized during the early 1980s in the USA with the aim of enhancing the level of US students in fields like science, technology, mathematics. The first time the acronym STEM was introduced was 2001, and since then has become "the buzzword among the many U.S. stakeholders who have heeded the call for creating better prepared high school and college graduates to compete globally" (Breiner et al., 2012).

For the STEM Task Force Report (2014), STEM education is far more than a "convenient integration" of its four disciplines, rather it encompasses "real-world, problem-based learning" that links the disciplines "through cohesive and active teaching and learning approaches" (p. 9). The report states that disciplines "cannot and should not be taught in isolation, just as they do not exist in isolation in the real world or the workforce" (p. 9).

Analysing the specialized literature emerges that in recent years there has been an increased commitment to interdisciplinary and transdisciplinary STEM integration (English, 2016). The traditional siloed subject teaching of STEM disciplines addressed by several authors (Bybee, 2010; Develaki, 2020; Millar, 2020) is being overcome by an integration of the various disciplines. Even though a unique definition of STEM Integration in the Science Education field must be made, Kelley and Knowles defined *Integrated STEM education* as "the approach to teaching the STEM content of two or more STEM domains, bound by STEM practices within an authentic context for the purpose of connecting these subjects to enhance student learning" (Kelley & Knowles, 2016, p.3).

Under the big umbrella of integrated STEM education "school disciplines are beginning to be integrated in an educational fruitful way" (Ortiz-Revilla et al., 2020, 857). Different epistemological views on STEM disciplines and their integration can have implications for general education for all and on the construction of future society as a whole (Ortiz-Revilla et al., 2020, 875). Due to its "young age", in the STEM integration field a lot of research remains to be conducted, such as the complexity of making crosscutting STEM connections, inadequate teacher knowledge incorporating all STEM fields, and the lack of materials and instructional and assessment support and guidance (Develaki, 2020). Educational researchers indicate that teachers struggle to make connections across the STEM disciplines (Kelley & Knowles, 2016. Secondary teachers have expressed difficulty in using frameworks from other disciplines or other teachers and felt that STEM curricula may be inflexible and not be able to impart meaningful learning (Margot & Kettler, 2019). Also, teachers expressed the belief that the availability of a quality curriculum would enhance the likelihood of success of STEM initiatives (Margot & Kettler, 2019). In their words, "Teachers need quality curriculum that aligns with district and state guidelines and includes formative assessment techniques teachers can use to assess their students' conceptual understandings" (Margot & Kettler, 2019, p.14).

1.4 The IDENTITIES Project

A core idea of the project IDENTITIS is that, in order to develop interdisciplinary pathways that crossboundaries between disciplines, it is essential to maintain and value the identities and peculiarities of each of them. IDENTITIES (*Integrate Disciplines to Elaborate Novel Teaching approaches to InTerdisciplinarity and Innovate pre-service teacher Education for STEM challenges)* is an ERASMUS + project coordinated by the research group in Physics Education of Bologna. In this project, indeed, the partners assume that the search for the meaning of interdisciplinarity cannot ignore the meaning of disciplines and their epistemological identities. The project IDENTITIES started in September 2019 (https://identitiesproject.eu). Together with the group of Bologna there are other 4 different partners coming from Italy (University of Parma), France (University of Montpellier), Spain (University of Barcelona) and Greece (University of Crete). The main goal of the project is to build innovative and transferable teaching modules and courses to be used in contexts of pre-service teacher education (e.g., curricula in Physics Education, Mathematics Education or Computer Science Education within master's degree courses). The central theme of the modules is interdisciplinarity in STEM fields, with a focus on the links and interweaving between physics, mathematics, and computer science.

The main objective of this thesis, realized in collaboration with the research group in STEM education of the University of Crete under the supervision of Prof. Dimitris Stavrou and Prof. Kostas Tzanakis, is to contribute to the IDENTITIES project by creating blended activities addressed to pre-service teachers on the specific topic of the Special Theory of Relativity (STR). These activities are based on the analysis carried out in the third chapter of the original texts of STR by Lorentz, Poincaré, Einstein and Minkowski (written between 1904 and 1908), aimed at recognizing the interplay between mathematics and physics implemented in the four papers. The analysis has been realized by applying the interdisciplinary lenses of the 'Boundary Crossing and Boundary Object' research framework developed in 2011 by Akkerman & Bakker (2011).

These activities are planned to be part of an IDENTITIES module regarding the crisis of Newtonian classical mechanics and the birth of non-Euclidean Geometries that will be implemented next year in a pre-service teacher education course. The activities have been designed with the goal of exploiting Special Relativity as a context to develop interdisciplinary skills and with the aim of understanding the process behind the development of the theory, the approach, and the image that the authors themselves had towards the discipline and in particular the different role that each author has given to Lorentz Transformations (LT).

Chapter 2 - Literature Review of Teaching and Learning STR in Physics Education and Mathematics Education

2.1 Introduction

In this chapter a structured review has been made on studies concerning the teaching and learning of Special Theory of Relativity (STR) in High School and University in Physics Education Research (PER) and Mathematics Education Research (MER) fields. The purposes of this review are to identify what are the main research topics, methodologies and research techniques related to STR in PER and MER, what are the main difficulties encountered by students and teachers in addressing the topic and what are the less considered aspects of this issue. A total of 55 papers were analysed, 49 in the PER field and 6 in the MER field.

STR was chosen as the area of the analysis for several reasons. STR has been a paramount field for developing theoretical reasonings on Conceptual Change (Hewson, 1982; Posner et al., 1982; Levrini & diSessa, 2008), i.e., for developing theories of learning in science education and learning science.

Due to the fact that different scientists coming from different fields (like physics, mathematics, chemistry, philosophy…) have contributed to the development of STR, this topic also offers the possibility of reasoning about the Nature of Science and the processes necessary for the development of scientific theories. Moreover, STR can be considered a transition topic, from classical to modern physics, and this role, together with the fact that it is usually taught in the last part of school Curriculum, gives special importance to the theme.

STR can be indeed considered a steppingstone towards the discoveries of the XX century.

All these reasons give STR a key role in understanding the scientific revolutions that have occurred since then, and so it is fundamental to understand how this theory is taught in High School and University, what are the difficulties that students encounter while studying these topics and how the teaching of this topic is changing due to new technologies and new pedagogic and epistemological approaches.

In the second section of this chapter the methodology of the review is described, while in the third and fourth sections the review is presented. The results are then discussed in the fifth section, together with the research questions that come out of the review and that will be addressed in the third chapter.

2.2 Methodology of the Review

Randolph's (2009) guidelines were followed to produce a review aimed to answer three questions:

1. What are the main strands focused on teaching and learning STR in the PER and MER fields? What are the most relevant papers for each strand?
2. What are the main difficulties students and teachers face in addressing STR?
3. What are the least addressed aspects in the literature?

To realize this review papers, conference papers and chapters of handbooks have been analysed. Since the literature concerning STR is very wide, only documents written in English that addressed the STR from an educational point of view have been considered.

To collect data an electronic search of academic databases on Internet have been realized, followed by a cross-reference search with other papers and reviews found, until a point of saturation was reached. The collection of data has been made in two different periods, December 2020, and May 2021.

This review aims at being exhaustive with purposive samples. A mixed-methods analysis have been performed: to briefly present the contents of every study, Title, Abstract, Introduction and Conclusions were analysed, with the aim of finding the main research themes in the literature. Intentions, Methodologies and Results (or Implications) have been then presented for every article following a chronological order adopting a narrative technique.

Subsequently, a more in-depth analysis of the most relevant papers for each research topic was carried out. To define which are the most relevant papers the chronological order and the cross-reference confrontation were considered.

Other techniques of analysis could have been used in order to obtain more general and complete results, but this is beyond the scope of this review, which is to answer the research questions highlighted above.

2.3 Teaching/Learning STR in PER

STR was and still is a case study of theoretical elaboration in Science Education and in the Learning Sciences, not only from a teaching perspective, but also in a more theoretically oriented research perspective. This topic has been analysed from several different aspects throughout the years, starting from the 1980s to the present day, and allowed for the development of theories of learning that have been fundamental also to the development of the PER as we know it today. To answer the research questions presented above, an attempt was made to identify the most recurring research themes in the literature on PER.

2.3.1 Data collection in PER

In order to create a solid database on STR in the PER framework, different methods of investigation have been used:

1. At first, research for keywords was realized. The keywords used in this phase were "teaching special relativity" and "learning special relativity" and the research has been made on the websites of the major journals in physics and in particular in PER, like:

    a. Physical Review - Physics Education Research (https://journals.aps.org/prper/)
    b. American Journal of Physics (https://aapt.scitation.org/journal/ajp)
    c. The Physics Teacher (https://aapt.scitation.org/journal/pte)
    d. Physics Education (https://iopscience.iop.org/journal/0031-9120)
    e. Physics Education Research – Conference Series (https://www.compadre.org/per/perc/)
    f. European Journal of Physics (https://iopscience.iop.org/journal/0143-0807)
    g. International Journal of Science Education (https://www.tandfonline.com/toc/tsed20/current)
    h. European Journal of Science Education (https://www.tandfonline.com/toc/tsed19/current)
    i. Science & Education (https://www.springer.com/journal/11191)
    j. Il Nuovo Cimento (https://www.sif.it/riviste/sif/jsearch)
    k. Journal of Physics – Conference Series (https://iopscience.iop.org/journal/1742-6596)

2. Secondly, a more general research has been made with the help of online databases for scientific publications, like:

    a. Google Scholar (https://scholar.google.com/);
    b. PER-Central (the Physics Education Research section of the sites ComPADRE, https://www.compadre.org/per/);
    c. ERIC (Institution of Education Sciences, https://eric.ed.gov/);
    d. Scopus (database of peer-reviewed literature from Elsevier, https://www.scopus.com/search/form.uri?display=basic#basic);
    e. Scitation (American Institution of Physics database, https://www.scitation.org/).

    The keywords used in this second stage were "teaching special relativity", "learning special relativity", "student difficulties special relativity" and "interdisciplinary special relativity". Also keywords like "conceptual change special relativity" were used, but their application did not lead to find further articles.

3. As a third and final research step, the papers have been cross-checked with other literature reviews, particularly with the literature review conducted by Alstein et al. (2020) in “Teaching and learning special relativity theory in secondary and lower undergraduate education: A literature review”. This review, as said before, was used to perform a tally of the identified literature. Even tough the objectives are slightly different, at least 10 papers relevant for my work were found in this review that did not emerged from the first two phases of the analysis.

2.3.2 Data Analysis in PER

At the end of the first two stages of the research a total of 45 papers, conference papers and handbooks’ chapters were found. After the third stage 10 more papers were added bringing the total up to 55. When the articles were analysed focusing on Title, Abstract, Introduction and Conclusions, 6 of them did not meet criteria (i.e., addressing STR from an Educational point of view). The final number of articles used for the review is therefore 49.

After identifying articles that met the criteria, the next step was to identify major research topics in the literature. To realize this categorization, a narrative summary of every article has been produced and then the various summaries were grouped together trying to identify the main theme or themes of each work. The main objective was to identify the leading research theme in each paper, although some of the papers analysed had more than one research theme.

This comparison between different works led to the definition of different categories, that are:

- Conceptual Change;
- Curricula Development;
- Digital Tools Development;
- History and Epistemology;
- Interdisciplinarity;
- In/Pre-Service Teachers Formation;
- Students’ Difficulties.

These categories have been chosen because they cover all the aspects dealt with in the various articles and contain the main objectives that the authors have set themselves in the realization of their work.

In Table 1 are reported all the articles divided for research themes. As said before, some of them are present in more than one category due to their multiple contents and objectives. In each category the articles are listed in chronological order, while the categories are listed in alphabetic order.

Table 1 - Categorization in Research Themes of STR’s papers in PER

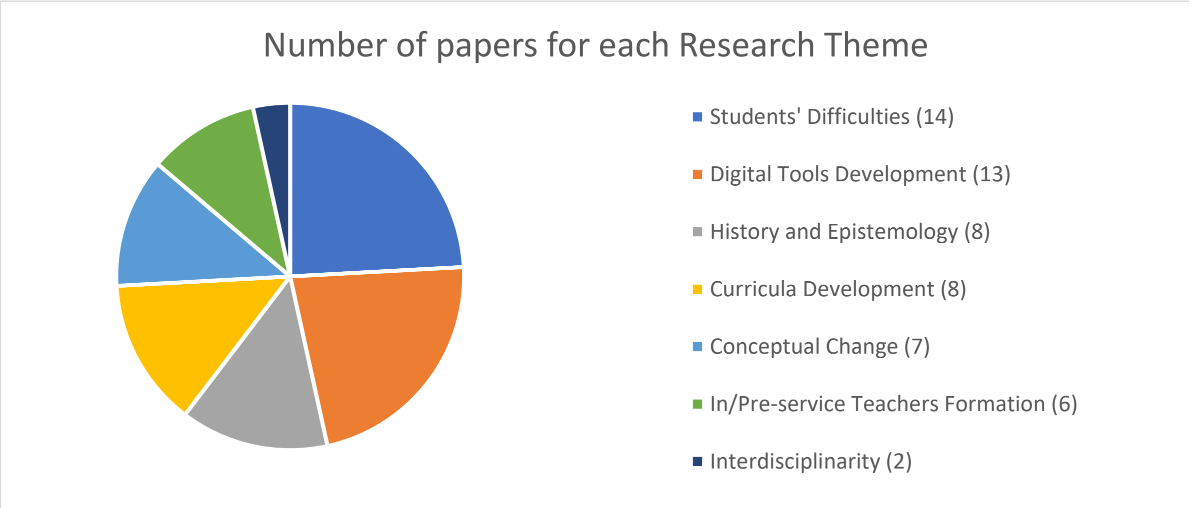
RESEARCH THEMES	PAPERS
Conceptual Change	1. Accommodation of a Scientific Conception: Toward a Theory of Conceptual Change (Posner et al., 1982) 2. A Case Study of Conceptual Change in Special Relativity: The Influence of Prior Knowledge in Learning (Hewson, 1982) 3. Special Theory of Relativity, Conceptual Change and History of Science (Villani & Arruda, 1998) 4. Student understanding of time in special relativity: simultaneity and reference frames (Scherr et al., 2001)

	5. The challenge of changing deeply held student beliefs about the relativity of simultaneity (Scherr et al., 2002) 6. How students learn from multiple contexts and definitions: Proper time as a coordination class (Levrini & diSessa, 2008) 7. Teaching the basic concepts of the Special Relativity in the secondary school in the framework of the Theory of Conceptual Fields of Vergnaud (Otero et al., 2015)
Curricula Development	1. Students' Understanding of the Special Theory of Relativity and Design for a Guided Visit to a Science Museum (Guisasola et al., 2009) 2. A constructive approach to the special theory of relativity (Miller, 2010) 3. A Teaching–Learning Sequence for the Special Relativity Theory at High School Level Historically and Epistemologically Contextualized (Arriasecq & Greca, 2012) 4. Teaching the basic concepts of the Special Relativity in the secondary school in the framework of the Theory of Conceptual Fields of Vergnaud (Otero et al., 2015) 5. Teaching Einsteinian physics at schools: part 1, models and analogies for relativity (Kaur et al., 2017) 6. "Einstein's Playground": An Interactive Planetarium Show on Special Relativity (Sherin er al., 2017) 7. A new model of special relativity and the relationship between the time warps of general and special relativity (Stannard, 2018) 8. Sensemaking in special relativity: developing new intuitions (Hahn et al., 2019)
Digital Tools Development	1. Teaching special relativity with a computer (Horwitz, 1994) 2. Teaching Special Relativity Using Physlets (Belloni et al, 2004) 3. Integration of information and communication technologies in special relativity teaching (Barbier et al., 2005) 4. Real Time Relativity: exploration learning of special relativity (Savage et al., 2007) 5. Student experiences of virtual reality: A case study in learning special relativity (McGrath, 2010) 6. Relativity in a rock field: A study of physics learning with a computer game (Carr & Bossomaier, 2011) 7. Novative Rendering and Physics Engines to Apprehend Special Relativity (Doat et al., 2011) 8. Learning Scenarios for a 3D Virtual Environment: The Case of Special Relativity (de Hosson et al., 2011) 9. Informal physics learning from video games: a case study using gameplay videos (Croxton & Kortemeyer, 2018) 10. Special Relativity in Immersive Learning (Chu et al., 2019) 11. Game Development for Teaching Physics (Kortemeyer, 2019) 12. Whiteboard Animation – A Tool for Teaching the Special Theory of Relativity (Chiriacescu, 2020) 13. Captain Einstein: A VR experience of relativity (Van Acoleyen, 2020)
History and Epistemology	1. Special Theory of Relativity, Conceptual Change and History of Science (Villani & Arruda, 1998) 2. The Substantivalist View of Spacetime Proposed by Minkowski and Its Educational Implications (Levrini, 2002) 3. Thought Experiments in the Theory of Relativity and in Quantum Mechanics: Their Presence in Textbooks and in Popular Science Books (Velentzas et al., 2007) 4. Approaches to the Teaching of Special Relativity Theory in High School and University Textbooks of Argentina (Arriasecq & Greca, 2007) 5. A Teaching–Learning Sequence for the Special Relativity Theory at High School Level Historically and Epistemologically Contextualized (Arriasecq & Greca, 2012)

	6. The Role of History and Philosophy in Research on Teaching and Learning of Relativity (Levrini, 2014) 7. Special Theory of Relativity in South Korean High School Textbooks and New Teaching Guidelines (Gim, 2016) 8. The 1895 Lorentz transformations: historical issues and present teaching (Provost & Bracco, 2016)
Interdisciplinarity	1. Using three-dimensional spacetime diagrams in special relativity (Dray, 2013) 2. Physics and Mathematics as Interwoven Disciplines in Science Education (Galili, 2018)
In/Pre-Service Teachers Formation	1. How physics teachers approach innovation: An empirical study for reconstructing the appropriation path in the case of Special Relativity (De Ambrosis & Levrini, 2010) 2. Exploring students' understanding of reference frames and time in Galilean and special relativity (deHosson, 2010) 3. Addressing pre-service teachers' understandings and difficulties with some core concepts in the special theory of relativity (Selçuk, 2011) 4. Prospective Teachers' Comprehension Levels of Special Relativity Theory and the Effect of Writing for Learning on Achievement (Yildiz, 2012) 5. Pre-Service Physics Teachers' Difficulties in Understanding Special Relativity Topics (Ünlü Yavas & Kizilcik, 2016) 6. Examining The Students' Understanding Level Towards the Concepts of Special Theory Of Relativity (Özcan, 2017)
Students' Difficulties	1. The use of the Principle of Relativity in the interpretation of phenomena by undergraduate physics students (Pietrocola & Zylbersztajn, 1999) 2. Student understanding of time in special relativity: simultaneity and reference frames (Scherr et al., 2001) 3. The challenge of changing deeply held student beliefs about the relativity of simultaneity (Scherr et al., 2002) 4. Integration of information and communication technologies in special relativity teaching (Barbier et al., 2005) 5. Modeling student thinking: An example from Special Relativity (Scherr, 2007) 6. How students learn from multiple contexts and definitions: Proper time as a coordination class (Levrini & diSessa, 2008) 7. Secondary Students' Understanding of Basic Ideas of Special Relativity (Dimitriadi & Halkia, 2012) 8. The Use of Thought Experiments in Teaching Physics to Upper Secondary-Level Students: Two examples from the theory of relativity (Velentzas & Halkia, 2013) 9. Relativity concept inventory: Development, analysis, and results (Aslanides & Savage, 2013) 10. Student difficulties in solving problems concerning special relativity and possible reasons for these difficulties (Tanel, 2014) 11. Special Relativity in Immersive Learning (Chu et al., 2019) 12. Sensemaking in special relativity: developing new intuitions (Hahn et al., 2019) 13. Students' pre-instructional reasoning with the speed of light in relativistic situations (Kamphorst et al., 2019) 14. Teaching and learning special relativity theory in secondary and lower undergraduate education: A literature review (Alstein et al., 2020)

These results can be discussed at two different levels. The first is merely numeric, and it is useful in order to understand what the categories are most studied among those analysed. The second one goes more in depth and focuses on the results of the studies, in order to reveal the state of the art of research on the teaching/learning of STR in Physics Education.

Graph 1- Number of papers for each category



From Graph 1 it is evident that the most analysed research theme is Student Difficulties, followed by Digital Tools Development, History and Epistemology, and Curricula Development. With respect to these strands of research, Conceptual Change and In/Pre-Service Teachers Formation are slightly less studied than the others, while Interdisciplinarity is by far the least addressed research theme.

The reviews shoes that the two main research themes of my Dissertation, In/Pre-Service Teachers Formation and Interdisciplinarity, appear the least investigated. In this sense, the work can be said to address a sort of blind spot and the final goal of developing a teaching course for pre-service teachers that show the deep interdisciplinarity inside STR has important elements of novelty in the PER literature.

2.3.3 Review’s results in PER

This section discusses the results of the review, analysing in detail the research themes one by one. the order of presentation of the categories depends on the number of articles for each of them, except for the Conceptual Change category, which because of its importance is presented first.

CONCEPTUAL CHANGE

In their famous paper Posner, Strike, Hewson and Gertzog (1982), STR is the context to elaborate the first model of conceptual change. They authors argue that, in order to comprehend fully the meaning and the consequences of physics knowledge, like STR, it is necessary to unpack the processes of assimilation and accommodation that characterize Piaget’s model of learning and adapt it to the specific field of physics. The model reproposes a Kuhnian way to model progress in science and describes four conditions that can trigger a process of learning as conceptual change: unsatisfaction for previous knowledge, intelligibility, plausibility, and fruitfulness of the new knowledge.

A study made by Hewson (1982) showed how important is the learner’s metaphysical commitments as fundamental components of existing knowledge to activate the four conditions for conceptual change and the effectiveness of an instructional strategy which explicitly addresses the epistemic novelties of STR to facilitate students to exploit their existing knowledge and accommodate the new one.

Villani and Arruda (1998) focused on how students can transform the content of the theory they have learned into stable knowledge. Usually, students do not use STR concepts, but keep their pre-SR ideas to interpret STR results. Their goal was to make the students aware of the existence and essential feature of a conceptual change in the history of science.

In 2001 and 2002 the Physics Research Group of the University of Seattle led by Scherr, Shaffer, and Vokos produced two paramount works for the development of PER and for the understanding of how students face conceptual change when studying topics such as STR. The first of the two works (Scherr et al., 2001) focuses on the understanding of time in special relativity, while the second one (Scherr et al., 2002) focuses on the challenge of changing deeply held student beliefs about the relativity of simultaneity. What emerges from these studies is that often students "construct a conceptual framework in which the ideas of absolute simultaneity and the relativity of simultaneity harmoniously co-exist" (Scherr et al., 2001, S24) and that "conceptual difficulties seemed to prevent students from answering correctly" (Scherr et al., 2002, 1247). In order to address these difficulties, the authors identified "the conceptual hurdles that hinder students from applying basic kinematical concepts to the complex situations encountered in special relativity" (Scherr et al., 2002, 1247) and designed tutorials that helped students in developing an understanding of the basic ideas of STR. These tutorials are described in the "Student Difficulties" category. They represent very useful materials to show how, in practice, conceptual change can eb fostered in teaching, although the design is very heuristic and not explicitly oriented to develop theories of conceptual change.

Levrini and diSessa (2008) used the theoretical framework of "coordination class theory", an evolving model of concepts and conceptual change, to describe the process undertaken by a class of students in understanding the concept of proper time. Analysing the discussion between the students, the authors showed that "proper time seems to have all the earmarks of a coordination class" (p. 14), in that the learning process exhibited by the students followed all of the learning mechanisms foreseen in the coordination class theory. Even though this theory is part of those theories called "humble", i.e., theories that try to explain a particular situation and do not pretend to generalize the results obtained, the authors showed how using this particular conceptual change theory was fundamental in order to understand how students, step by step, following the help of two teachers, reached a deep understanding of the concept of proper time. STR in this case was shown to be the ideal frame for the application of this theory of conceptual change.

Using Vergnaud's Theory of Conceptual Fields Otero, Arlego and Prodanoff (2015) developed a didactic sequence for the teaching of basic aspects of STR in High School. In their results the Principle of Relativity emerges as one of the most complex aspects of the theory, because most students were not able to adapt to the conceptual change required by STR. To solve this problem, they worked on conceptualization of the basics of the theory, using Galilean Relativity as a previous step to its generalization in STR. In order to understand fully STR students have to reject or at least accept the concept of Absolute Time, and to do that it is necessary to focus more and so conceptualize the basics of the theory, like the postulates.

STUDENT DIFFICULTIES

Pietrocola and Zylbersztajn (1999) designed a set of interviews aimed at exploring how 21 students coming from the first and last year of a Physics' University course changed their way of thinking from Classical Physics reasoning to a new theory like STR. The "clinical interviews were based on physical situations presented either in drawings or in simple sets with real equipment" (p. 263), and they were structured in order to make the students analyse "situations where mastery and application of the Principle of Relativity by the subjects could be evaluated" (p. 263). Results showed that none of the students mentioned explicitly the Principle of Relativity to explain the non-existence of change in the presented phenomena, while they did not see those situations as problematic and explained them using classical physics reasoning.

Scherr and colleagues (2001) realized an investigation of student understanding of the concepts of time, simultaneity and reference frames that focuses and illustrates how students reason with fundamental concepts of Special Relativity. Analysing the answers to three questions (that are Spacecraft, Explosion and Seismologist) and conducting interviews with each student, the authors found that the students failed to "recognize spontaneously that simultaneity is relative" (p. S26) and to "apply spontaneously the formalism of a reference frame in determining the time of an event" (p. S27). Still, the authors showed several beliefs among students, like that "events are simultaneous if an observer receives signals from the events at the same instant" (p. S28), that "simultaneity is absolute" (p. S29), and that "every observer constitutes a distinct reference frame" (p. S32).

In a successive work, the same research group (Scherr et al., 2002) described the development and assessment of instructional materials intended to improve student understanding of those concepts analysed in their previous work (Scherr et al., 2001), that are the concept of time in Special Relativity, the relativity of simultaneity, and the role of observers in inertial reference frames. Two tutorials have been developed in this work, i.e., "Events and reference frames and Simultaneity" (p. 1239). Results of the investigation reported that "many students who study special relativity at the undergraduate to graduate levels fail to develop a functional understanding. Even in advanced courses, students often do not recognize the implications of special relativity for our interpretation of the physical world" (p. 1247), but students that worked with the developed instructional materials "improved significantly in their ability to recognize and resolve some of the classic paradoxes of Special Relativity" (p. 1247).

Focusing on the possibility of teaching the basic ideas of STR to high school students, Dimitriadi & Halkia (2012) "investigate students' learning processes towards the two axioms of the theory (the principle of relativity and the invariance of the speed of light) and their consequences (the relativity of simultaneity, time dilation and length contraction)" (p. 2565). Even if students can cope with the basic ideas of STR, their study showed that they have difficulties with some conceptions that hinder a deep and meaningful understanding ot the theory, like the implic assumption of the existence of an absolute, privileged frame of reference, of fixed object's properties, and the independence of the spacetime relations among events from observers' point of view.

Velentzas & Halkia (2013) focused on the role of thought experiments and showed that they can help students in understanding and realizing situations that refer to an abstract world like the one described by STR. Using thought experiments students can grasp the deep meaning of physics laws and principles that require a high degree of abstract thinking. In particular, "the use of TEs in teaching the theory of relativity can help students realize situations which refer to a world beyond their everyday experience and develop syllogisms according to the theory" (p. 3026).

Aslanides and Savage (2013) developed a Relativity Concept Inventory (RCI) based on the template of the well-known Force Concept Inventory. The primary purpose of this study was "to provide an instrument for measuring changes in students' conceptual understanding of special relativity. In this role it would be administered prior to instruction as a pre-test, and after instruction as a post-test" (p. 1) and "to identify students' misconceptions" (p. 1). In the Inventory are present 24 multiple-choice questions addressing 9 concepts, that are: First postulate, Second postulate, Time dilation, Length contraction, Relativity of simultaneity, Inertial reference frame, Velocity addition, Causality, Mass-energy equivalence. Besides the fact that the Inventory showed some flaws, it showed "a gender bias that was not present in course assessment, similar to that reported for the Force Concept Inventory" (p. 1).

Tanel (2014) used mixed research methods with the aim to investigate student difficulties related to the theory of special relativity in 78 students enrolled in a Modern Physics course. What emerges from this study is that students "can express the changes that occur in length and time at speeds close to the speed of light. However, they have difficulties in applying their knowledge to solve problems about this topic" (p. 580). Because situations treated in STR are far from every day's experience, "their lack of experience makes it hard for them to believe the obtained results implied by the solutions of these problems and prevents them from cross-checking to see whether their solutions are correct" (p. 580).

Hahn and colleagues (2019) discussed student sensemaking about Special Relativity using visualization with spacetime diagrams and the development of rules of thumb (i.e., sentences that "serve as a footholds when solving problems in special relativity, like "proper time is the shortest time" (p. 196)). Analysing students' use of rules of thumb in written solutions to homework problems the authors found that students' use those rules with two purposes, i.e., as a means of orienting to the problem and as a means of reflecting on their answers.

A qualitative study on 15 students led by Kamphorst (2019) focused on students' pre instructional reasoning about the speed of light in STR. Results showed that the students use two different kinds of reasoning, that are reasoning with absolute speed of light or uniform speed of light relative to a certain reference frame. Using Event Diagrams as a tool to study how students reason about light propagation, their work showed that "many students do not only evaluate the speed of light relative to their preferred reference frame for uniform light propagation, but that they also are able to do this relative to other reference frames. Students are aware that light travels at a different speed as seen from these other frames, but most students do not spontaneously regard this as problematic. Some students did experience a conflict when comparing light propagation in several reference frames, which led them to change their reference frame for uniform light propagation" (p. 10).

A major part of these papers are described in the literature review realized by Alstein and colleagues (2020) regarding teaching and learning Special Relativity in secondary and lower undergraduate education. In the work they analysed 39 articles exploring the reported learning difficulties, teaching approaches and research tools. The analysis shows that "students at all educational levels experience learning difficulties with the use of frames of reference, the postulates of SRT and relativistic effects" (p. 1). To address these difficulties, that are well reported in the literature, the authors reported several teaching approaches, that they divided into four categories, i.e., focus on Thought Experiments, historical and philosophical contextualization, multimedia support, and student activities. A variety of teaching approaches, aiming at distinct learning objectives, have been reported to help students overcome learning difficulties. The main learning objectives are "to foster conceptual understanding; to foster understanding of the history and philosophy of science; to gain motivation, confidence and attitude toward learning SRT. Although there is inevitably some overlap, teaching approaches that rely heavily on TEs mainly aim at the first objective, HPS-contextualized teaching approaches aim at the second objective, while multimedia supported teaching approaches tend to focus more on the third objective" (p. 13).

DIGITAL TOOLS DEVELOPMENT

Horwitz et al. (1994) designed a computer-based Relativity Laboratory (RelLab) "to lead high school or introductory college physics students to a qualitative understanding of Special Relativity" (p. 92). This software is "an open-ended interactive "construction kit" with which students set up and perform a wide variety of *gedanken* experiments involving systems in motion at speeds from the everyday to the relativistic" (p. 92). As a result, students that were bored during the standard lesson gave evidence of enjoying the process and learned better in this way.

With the goal of covering the inadequacies that classical pencil-and-paper exercises alone can have in helping students understand abstract STR ideas, Belloni et al. (2004) developed Physlet-based activities incorporating the results from Scherr's works (2001; 2002). These activities focused on "visualizing the relativity of simultaneity, length contraction, time dilation, and spacetime diagrams" (Belloni et al., 2004, p. 8).

Barbier et al. (2005) developed multimedia content aimed at enhancing the learning process of undergraduate students in the study of Special Relativity. In the authors' opinion the integration of

Information and Communication Technologies (ICTs) in the teaching of STR "may bring multiple and complementary methods for introducing either difficult or abstract counterintuitive concepts" (p. S13). In their work they concentrated on "animated scenarios built to carefully emphasize the contribution of experiment and intuition to generate the physical theory of special relativity" (p. S21), categorizing their developments into two types of ICT scenarios, that are real experiments and thought experiments.

Savage et al. (2007) designed a computer program named "Real Time Relativity" where students can fly at relativistic speeds in a simulated world with several objects like planets or clocks. This project has been made with the goal of highlighting the "counterintuitive and spectacular optical effects of relativity … while systematic exploration of the simulation allows the user to discover relativistic effects such as length contraction and the relativity of simultaneity" (p. 1).

In a subsequent work, McGrath et al. (2010) implemented and evaluated the same software from Savage et al. (2007), i.e., Real Time Relativity. Moreover, they designed learning activities based on this software performed over four semesters in two Australian Universities in physics' laboratory courses. Through the use of the simulations, students "were able to develop visual models of the effects of special relativity" (p. 867), and the model "aided their understanding and enabled the students to see the topic as less abstract" (p. 867).

A game based on Asteroids, named Relativistic Asteroids, was designed, and developed by Carr & Bossomaier (2011) with the aim of providing "an open environment for players to interact with the physics, eliciting their conceptions as they explore and experiment, and promoting comparison and reflection" (p. 1049). The results of different tests showed that the game "is effective at introducing some concepts of special relativity to learners, aligned with the Australian Higher School Certificate (HSC) high school physics curriculum.

Addressing the difficulties that students encounter when facing Special Relativity, like going beyond mathematical equations or understanding the deep implications of the theory, Doat et al. (2011) developed a "framework designed to merge advanced 3D graphics with Virtual Reality interfaces in order to create an appropriate environment to study and learn relativity as well as to develop some intuition of the relativistic effects and the quadri-dimensional reality of space-time" (p. 9). To do so they designed and implemented an easy-to-use game-like application based on a billiard, with the special feature of implementing "the 4D nature of space-time directly at the heart of the rendering engine" (p. 9) and developing "an algorithm allowing to access non-simultaneous past events that are visible to the observers at their specific locations and at a given instant of their proper time" (p. 9).

A similar research has been conducted within the EVEILS project (French acronym for Virtual Spaces for the Education and Illustration of Science) with the help of an immersive environment named CAVE (Cave Automatic Virtual Environment) by de Hosson et al. (2014). In their work they undertook a lexical analysis with the aim of searching specific conceptual elements considered as key points for the user's understanding. These elements are the incoming of light in the eye, the finite nature of the speed of light, the distance between the pucks and the user, the object discretization (a set of points as punctual sources of photons), and the geometrical relativistic effects explained by the Lorentz transform" (p. 383).

A case study has been made by Croxton & Kortemeyer (2018) based on the use of the game "A Slower Speed of Light", a video game designed to teach aspects of Special Relativity. In this video game abstract concepts are made accessible in human-scale environments and first-person view. In their work they analysed 20 voluntary made gameplay videos based on the game on YouTube from several points of view: awareness, knowledge or understanding; engagement or interest; attitude; behaviour and skills. This analysis of these "voluntarily produced gameplay videos with their free-flowing think-aloud monologue and associated viewer comments can provide valuable additional insights into informal learning through games, particularly in terms of informing game designers and educators for upcoming versions of their games" (p. 10).

Chu and colleagues (2019) developed, implemented, and evaluated an "interactive, real-time, and real-scale virtual relativity application used to understand the theory of Special Relativity" (p. 16). Their study showed that "visual reinforcement of objects under Lorentz contraction as well as several task to experience the twin paradox improved the knowledge and skills in special relativity" (p. 27), and therefore "special relativity simulation in virtual reality provides a high level of immersion and enjoyment and has a significant positive learning outcome" (p. 27).

Kortemeyer (2019) in his work presented two video games developed to teach physics concepts. One of these is "A Slower Speed of Light". Using the same concept of Sherin and colleagues (2017), in the game "players encounter Special Relativity in an environment where the speed of light is lowered to everyday velocities" (p. 1). As a result, they found that "gameplay can be highly engaging, but needs to be structured and include explanations in order to address learning goals" (p. 5).

Using an approach called "whiteboard animation", that is presenting notions with the help of animation taking place on the surface of a whiteboard, Chiriacescu and colleagues (2020) developed a conceptual approach where Special Relativity basic concepts were presented using "basic notions, some logical thinking and an audio-visual way to display it" (p. 5). Even if this approach is considered maybe too simplistic by the authors, they agree on the fact that with their tools students have a "better motivation that finally leads to a more effective learning process" (p. 5).

Van Acoleyen & Van Doorsselaere (2020) produced a Virtual Reality movie where who is watching is moving on a boat that travels in a world where the value of speed of light is low, as in Kortemeyer (2019) and Scherin et al. (2017). The movie presents the basic STR's effects, light aberration and also the relativistic Doppler Effect and has been experimented both as a science communication and outreach tool and an educational tool in relativity courses at the undergraduate university level.

HISTORY AND EPISTEMOLOGY

Levrini (2002) in her paper discussed the debate between substantivalism (a physical object endowed with substantiality) and relationalism (a set of relations constructed by human reason to organise the factual world) and presented a reconstruction of Minkowski's 1908 view of Spacetime and argued how his interpretations "can be considered a substantivalist interpretation of SR and, consequently, to what extent it represents the key to building a consistent substantivalist line running from Newtonian mechanics to GR" (p. 602).

Velentzas, Halkia and Skordoulis (2007) investigate the presence of Thought Experiments (TEs) related to STR in physics textbooks and in generic books and their use in introducing modern physical theories of the 20th century. In this work it is shown that TEs can have different roles (introduction, methodological, attraction) and they can trigger students' interest and act as educational material to acquaint them with concepts and principles of the physics theories of the 20th century which at a later time they will study in depth (p. 367).

An analysis of the representation of the special theory of relativity in the most used books in the Argentine Republic's schools has been made by Arriassecq and Greca (2007). They showed an "inefficiency of the didactic material available for the teachers to approach, from a contextual perspective, the introduction of the SRT in high school/polimodal cycle" (Ariassecq & Greca, 2007, 82) and expressed the need for the elaboration of didactic material for students and teachers aimed to introduce "physics contents from a contextualized point of view - conceptually appropriate and motivating – to give a meaningful learning to students" (p. 82).

In a subsequent 2010 paper the same authors developed a teaching learning sequence for the STR at high school level historically and epistemologically contextualized. In their opinion "this approach makes it possible to appreciate the contributions made by several historical characters towards the creation of a new scientific conception, even though later they were not successful, or did not receive the recognition of their peers at the time they developed their ideas" (Ariassecq & Greca, 2010, 848). Moreover, they stressed the importance of having a conceptual discussion in the teaching of SRT about those concepts that underlie the theory and therefore give meaning to an approach that if not could not be called epistemological and historical (Ariassecq & Greca, 2010).

A useful literature review concerning the role of history and philosophy in teaching and learning Special Relativity has been carried out by Levrini (2014). The review showed that "the research strand concerning students' difficulties in learning special relativity is well developed and shared results have been achieved" (p. 178), while there are "deep unsolved research problems [concerning] the design of teaching materials and the dissemination of good practices through teacher education" (p. 178). According to the author, further studies are needed with the goal of outlining new criteria for developing a collaborative relationship between faculty, students, designers, and materials.

Analysing South Korean high school textbooks Gim (2016) showed two main flaws in how STR is presented to students: "the textbooks' contents present historically fallacious backgrounds regarding the origin of this theory because of a blind dependence on popular undergraduate textbooks, which ignore the revolutionary aspects of the theory in physics. And second, the current ingredients of teaching this theory are so simply enumerated and conceptually confused that student are not provided with good opportunities to develop critical capacities for evaluating scientific theories" (p. 575). He then outlined an alternative way for introducing STR's postulates, "emphasizing that the history of science is very useful for grasping the origins as well as the meanings of the two postulates of the STR" (p. 604).

In their work Provost and Bracco (2016) showed the pedagogical interest of the 1895 Lorentz Transformations (LT) for the teaching of Special Relativity. This version of the transformations, that differ from the 1904 ones discussed in the third chapter for their simplicity, are for the authors easier to understand in that there are not the γ factor or hyperbolic functions, but they keep the necessary aspects to address the standard issue of Special Relativity. For them, "the possibility of avoiding the technical aspects of the 1904 LT while keeping the spirit of relativity makes the approach of SR through the 1895 LT an instructive one. (p. 2).

CURRICULA DEVELOPMENT

In their paper Guisasola and colleagues (2009) described "the curricula design used as a learning tool in school visits to a science museum" (p. 2085). In this paper they discuss the way in which they built a teaching sequence based on an exhibition on 'A century of the Special Theory of Relativity' realized in Spain with the attempt of bridging the gap between formal teaching and the exhibition visit. The teaching sequence is based on the contextual model of learning by Falk and Dierking and has been designed "taking into account the possibilities that are offered by each situation in school and in the museum. So, the pre-visit and post-visit activities are based more on analysing and discussing carefully selected situations involving STR, whilst the activities in the museum aim to provide students with an opportunity to experiment, provide meaning and interest to the questions raised, and let them search for information" (p. 2088).

Miller (2010) in his work tried to derive STR results with a constructive approach, based on the dynamics of the problem, in order to solve the difficulties encountered by students when STR is taught only from a purely kinematical point of view. In his paper he starts with the description of a physical law, and from that he "investigates the behaviour of measuring rods and clocks" (p. 634). Including a constructive or dynamical approach is for the author an approach that enriches students' understanding of the subject.

Kaur and colleagues (2017) developed a project, named Einstein-First, aimed at introducing modern Einsteinian concepts of space and time, quanta, and gravity at an early age. In order to do that they focused on the development of models and analogies, considered the key for implementing those kinds of curricula in such an early phase (Year 6 primary school). To introduce Special Relativity, they focused on the "universality of the speed of light and how it is enforced" (p. 9). The activities introduced showed an "increase in students' enjoyment and attitudes toward the sciences, leading to more positive learning outcomes" (p. 11).

Sherin and colleagues (2017) developed an interactive planetarium show aimed at showing Special Relativity's effects in an everyday scenario, in that usually Relativity is not treated in planetarium shows for the general public due to the difficulties related to the topic and to the visualization of STR's effects. To solve this problem, they "simulate a universe in which the speed of light is slower, so that "everyday" speeds become relativistic" (p. 550). To do so they used the OpenRelativity library and built a fully immersive environment, where participants were able to look around during their experience, observing the environment around them and focusing on the differences between their front, their side and behind them.

Stannard (2018) developed new models and functional analogies for teaching STR in high school, with the goal of merging Special and General relativity together as one topic in a way that was logical and consistent. The paper introduces a "simple model to illustrate the time warping consequences of Special Relativity and General Relativity" (p. 8) and shows how STR is intrinsically embedded in GR.

IN/PRE-SERVICE TEACHER FORMATION

In the paper of De Ambrosis and Levrini (2010 "the problems related to innovation are addressed focusing on the phase during which teachers analyse an innovative proposal in the perspective of designing their own teaching paths" (p. 2). In this study the authors focus on the process that teachers face in order to "appropriate the approach to relativity proposed in the text "Spacetime Physics" by Taylor and Wheeler and to compare it with more traditional ones in the perspective of designing their own path for the class activities at the secondary school level" (p. 2). The root of the problem concerning innovative proposals can be found and addressed in the attempt to appropriate them even before trying to apply it, in that teachers attempt usually to transform and tune the proposals in their own style but sometimes they are not able to do that. In this study the authors reach positive results, as they focused and with several activities fostered the appropriation process of the teachers.

A study conducted by de Hosson, Kermen and Parizot (2010) administered to 100 prospective physics teachers revealed a deep lack of understanding of both concepts of reference frame and event. Also, these students showed misunderstandings towards the concept of simultaneity. For them this lack of understanding depends on the fact that students have difficulties in switching from the classical kinematics framework to the relativistic one. Moreover, in their study they highlighted that there were no evident differences between prospective teachers that attended STR courses and those who did not, showing the inefficacy of those courses.

Using questionnaires and interviews on 185 pre-service teachers Selçuk (2010) found that at different academic levels they have specific difficulties with proper time, proper length, time dilation, mass, and relativistic density concepts. These difficulties stayed even after instruction in STR. The questionnaires are a useful instrument to determine the students' understanding and existing misconceptions about the issue.

Studies conducted on pre-service teachers showed that the comprehension levels of STR were low. In the attempt to solve this understanding Yildiz (2012) used the writing for learning activities technique and showed how this approach helped students achieve a deeper understanding of critical aspects of the theory and was effective in increasing the students' comprehension levels of STR.

In Ünlü Yavas and Kizilcik (2016) work is shown that students are biased against relativity subjects and consider them to be difficult. Also, problems related to the mathematical aspects of the theory, the determining of the reference system and the change from classical to relativistic physics made the learning process challenging for them.

Interviewing fourteen pre-service physics teachers Özcan (2017) found that in order to answer some particular questions regarding key STR concepts students mixed scientific and unscientific knowledge to make explanations, and this inappropriate use of these knowledge structures led to unscientific explanations and misunderstandings related to concepts like reference frames, time dilation and length contraction.

INTERDISCIPLINARITY

Dray (2013) in his work focuses on the use of geometric reasoning with three-dimensional spacetime diagrams instead of algebraic manipulations using three-dimensional LT. These examples, that are the "rising manhole" paradox, the "moving spotlight" problem, and Einstein's light-clock derivation of time dilation, are useful in that they show directly the "counterintuitive effects of transforming velocities between reference frames" (p. 596) and provide "new insight into a standard thought experiment about the Lorentzian inner product of special relativity" (p. 596).

The deep connection between mathematics and physics has been analysed by Galili (2018). In this work, which comprehensively and thoroughly covers several aspects of this interplay, the author describes Special Relativity as an illuminating example of the intimate relationship between the two disciplines. Both the differences between the approaches from Einstein and Lorentz, or the geometrical representation of the theory, can be used to stress the different roles played by the disciplines. In his opinion then, there is a need in upgrading school curricula focusing on the specific nature of each discipline and their "specific agenda, ontology, epistemology, and values" (p. 32).

Regarding the interdisciplinarity between mathematics and physics, a substantial body of work was produced in the 2015 Science & Education special issue. Among the many papers published in this issue, there are some that are really interesting for the analysis of the relation between the disciplines from multiple points of view, even though they do not directly address Special Relativity (e.g., Brush, 2015; Karam & Krey, 2015; Kragh, 2015).

2.4 Teaching/Learning STR in MER

Similar work to that conducted in Section 2.3 was carried out with the goal of analysing the literature inherent in the teaching and learning of STR in the MER field.

However, being that STR is not a topic generally covered by math curricula in high schools a significantly smaller number of articles were found.

Some of these papers focus on the interdisciplinary aspect of STR, while others focus primarily on the geometric aspects of the theory, developing methodologies that base their reasoning for obtaining the results of the theory on the concept of group of transformations.

2.4.1 Data Collection in MER

Following the same approach used in the previous section, websites of the major journals in mathematics and in particular in mathematics education were analysed. These journals are:

- Educational Studies in Mathematics (https://www.springer.com/journal/10649);
- For the Learning of Mathematics (https://flm-journal.org/);
- Research in Mathematics Education (https://www.tandfonline.com/toc/rrme20/current);
- European Journal of Mathematics (https://www.springer.com/journal/40879);
- ZDM – Mathematics Education (https://www.springer.com/journal/11858);
- Journal for Research in Mathematics Education (https://www.nctm.org/publications/journal-for-research-in-mathematics-education/);
- Journal of Research in Mathematics Education (https://hipatiapress.com/hpjournals/index.php/redimat);
- Journal of Mathematics Teacher Education (https://www.springer.com/journal/10857);
- International Journal of Mathematical Education in Science and Technology (https://www.tandfonline.com/toc/tmes20/current);
- The Mathematics Teacher – National Council of Teachers of Mathematics (https://www.jstor.org/journal/mathteacher).

In a second phase other databases were used to find other papers, like Google Scholar or MathsEduc (https://www.mathseduc.com/journals/).

The keywords used in the first phase were “interdisciplinarity special relativity” and “interdisciplinary special relativity”. In the second phase, a more generic search was made using “special relativity” as keywords, but the results were almost the same.

2.4.2 Data Analysis in MER

Due to the small number of the papers found, shown in Table 2, a systematic categorization of the papers was not possible. All the papers found focus on the role that group transformations can have in the explaining of STR fundamental aspects, except for the paper from Fiore (2000), in which the author uses a narrative approach to introduce the basic aspects of STR. With regards to the main research strand found, the works made by Tzanakis are the most relevant in that they cover several aspects of the theory from an interdisciplinary point of view (mathematical, physical, and historic).

Table 2 - Categorization in Research Themes of STR's papers in MER

RESEARCH THEME	PAPER
Group Transformations' Role in STR	1. Unfolding interrelations between mathematics and physics, in a presentation motivated by history: two examples (Tzanakis, 1999) 2. Integrating the Close Historical Development of Mathematics and Physics in Mathematics Education: Some Methodological and Epistemological Remarks (Tzanakis & Thomaidis, 2000) 3. On the relation between Mathematics and Physics in undergraduate teaching (Tzanakis, 2002) 4. Mathematics & physics: an innermost relationship. Didactical implications for their teaching & learning (Tzanakis, 2016) 5. La 'Rivoluzione' di Felix Klein (Speranza, 1997)
Narrative approach (?) to STR	1. An Out-of-Math Experience: Einstein, Relativity, and the Developmental Mathematics Student (Fiore, 2000)

2.3.3 Review's results in MER

Among the papers found in the literature, the works carried out by Tzanakis (1999, 2002, 2016; Tzanakis & Thomaidis, 2000) are particularly detailed in the analysis of the interrelation between mathematics and physics in Special Relativity. Special Relativity is nowadays a standard in physics curricula, both at high school and undergraduate level, while in Mathematics curricula it is a topic treated sometimes in undergraduate or postgraduate level. In author's opinion "basic aspects of both GR [General Relativity] and SR [Special Relativity] that played an important role in the development of new Mathematics and enhanced the development of our understanding of physical phenomena, can be presented at a much earlier stage as an illustration of this new Mathematics and their place in the scientific edifice" (Tzanakis, 2002, 1452). Using a history-pedagogy-mathematics/physics (HPM/Ph) perspective, Tzanakis has analysed and explored the links between the two disciplines from a historical and epistemological point of view.

To describe this interplay the author formulated three main theses (Tzanakis, 2016):

- Thesis A) "Mathematics and physics should be conceived (hence, taught and learnt) both as the result of intellectual enterprises and as the procedures leading to these results. Knowledge gained in their context has an evolutionary character; by its very nature, historicity is a deeply-rooted characteristic" (p. 4)
- Thesis B) "Throughout their historical development from antiquity to the present, mathematics and physics have been evolving in a close, continuous, uninterrupted, bidirectional, multifaceted and fruitful way" (p. 4)
- Thesis C) "Mathematics and physics as embodiments of general attitudes in regard to the description, exploration, and understanding of empirically and/or mentally conceived objects, are so closely interwoven, that any distinction between them is related more to the point of view adopted while studying particular aspects of an object, than to the object itself" (p. 5).

To support these theses, the author describes several examples that allowed him to analyse the deep relationship between the disciplines (Tzanakis 1999, 2002, 2016). STR is one of these examples:

- Measurement of distances of inaccessible objects;
    - Eratosthenes's measurement of the earth's circumference;
    - Aristarchus' "lunar dichotomy" to measure the earth-sun-moon relative distances;
    - Copernicus' method for measuring inner planets' relative distances from the sun.
- Rotations, Space-time, and the Special Theory of Relativity;
- Differential Equations, (Functional) Analysis and Quantum Mechanics;
- Newton's gravitational law derived from Kepler's laws;
- The physical origin of many basic concepts and theorems of the theory of Dynamical System and of Ergodic Theory.

As a result of the analysis, the author says that "it is impossible to deeply understand either mathematics or physics without being sufficiently aware of their interconnections and mutual influence" (Tzanakis, 2016, 24) and also that "taking into account their interrelation is beneficial for teaching and learning either discipline" (Tzanakis, 2016, 24).

As far as Special Relativity is concerned, Tzanakis (1999, 2002, 2016) explores, first, the historical origins of the theory, describing the different approaches used by Einstein, Lorentz, Poincaré and Minkowski in formulating the theory. Then he outlines a didactical implementation aimed at obtaining LT using a reasoning based on plane rotations in 2D-analytic geometry. To do so, he represents Euclidean Rotations with orthogonal transformations with matrix parametrized by the rotation angle φ

$$R\varphi = \begin{vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{vmatrix}$$

and, then he represents LT by pseudo-rotations with matrix parametrized by φ, with tan φ =v/c

$$R\varphi = \begin{vmatrix} \cosh\varphi & -\sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{vmatrix}$$

At this point, the didactical approach foresees an activity aimed to guide the students to use the group structure of the transformations to obtain the main results of the Special Theory of Relativity, like length contraction or velocity addition; also, students are shown to recognise that the conformal transformations are isometries of the Minkowski (pseudo)-distance. In such a way also the relativistic law of velocity can be obtained, by reasoning on the group's properties of the Lorentz' Transformations (or also directly from Maxwell's Equations).

Speranza (1997), in his epistemological analysis of the "Erlangen program" made by Felix Klein in 1872 (Klein, 1893), underlines the role played by the Erlangen Program in inspiring the STR. Following a period in the 19th century when many new geometries were being born, with his program Felix Klein overcame the plurality of the various geometries in a higher order perspective, defining each geometry on the basis of the study of concepts, properties, and invariant relationships due to the transformations of a group (Speranza, 1997). Indeed, the space-time from Einstein and Minkowski deeply depends on this new way of understanding geometries, in that the SpaceTime is a four-dimensional space structured by the LT, recognized as a group by Poincaré in 1905 (Poincaré, 1906).

Other works focused on the importance in history of Klein's program but none of them analysed the didactic implications of his work on Special Relativity.

The influence of the Erlangen Program on the studies conducted at that time by several physicists and mathematicians (such as Henri Poincaré) was fundamental for the following development of both the fields, as stated in many works (e.g., Hawkins, 1984; Rowe, 2018; Weigand et al., 2014).

Using a different narrative with respect to the other works described here, Fiore (2000) introduces basic Special Relativity's concepts from the perspective of a mathematics teacher. Using Einstein's anecdotes and examples from everyday life with the help of images and graphs, the author succeeds in a few passages to outline the basic features of the theory and to describe the main applications and the various findings of the theory, such as the study of muons and cathode ray tubes.

2.5 Final commentary on the review

A total number of 55 papers have been presented in this review, 49 in the Physics Education Research field and 6 in the Mathematics Education Research field. The results of this review show that STR, even if it is a typical physics topic, is a relevant example for interdisciplinarity and for reflecting on mathematical concepts, such as geometric representations and transformations.

The research questions attempted to be answered with this review are:

1. What are the main strands focused on teaching and learning STR in the PER and MER fields? What are the most relevant papers for each strand?
2. What are the main difficulties students and teachers face in addressing STR?
3. What are the least addressed aspects in the literature?

In order to answer the first, during the review 7 different strands were found in the literature in the PER (Student Difficulties, Digital Tools Development, History and Epistemology, Curricula Development, Conceptual Change, In/Pre-Service Teachers Formation, Interdisciplinarity). The most addressed strands are Student Difficulties and Digital Tools Development, while the least addressed is Interdisciplinarity. As far as the MER field, the biggest strand is the ones that focuses on the geometrical aspects of STR from an interdisciplinary point of view, proving how group transformations are useful for the understanding of the basic aspects of the theory.

The Conceptual Change strand has been emphasised since it played a foundational role for the research in PER and, within this strand, STR was a central theme. For this reason, STR holds a privileged role among the topics studied in school and university curricula as it has allowed the emergence of learning theories in science education, thus in a pure research perspective.

Student Difficulties is evidently the most addressed research theme, in that students usually face several challenges in understanding the core of the theory. Works as the ones from the Seattle research group in PER (Scherr et al., 2001, 2002) or Velentzas and Halkia (2013) and the literature review from Alstein and colleagues (2020) showed that students have problems in understanding STR's effects since they are far from their everyday experience and often counter-intuitive. Also, the students often tend to make to coexist internally a double conception of the studied phenomena, absolute and relative, due to the not acceptance of what is going to be studied.

Several methods and approaches have been developed with the attempt of solving these problems. Thought experiments are a useful tool that helps students in imaging what are the effects of theory's postulates (Velentzas & Halkia, 2013).

Furthermore, a very wide strand focused on solving difficulties related to this problem is the Digital Tools Development one. Using games and software based on 3D graphics or Virtual Reality, many authors (Barbier et al., 2005; Savage et al., 2007; Kortemeyer, 2019; Van Acoleyen, 2020) showed how the possibility to assist "live" to relativistic phenomena and to touch with hand these realities help students to have a deeper understanding of the theory.

Other studies focused on Curricula Development and History and Epistemology, two fields with several connections between them. The most relevant results here are the ones of Levrini (2014), Arriassecq and Greca (2007, 2010). What is stressed in these studies is that there are "deep unsolved research problems [concerning] the design of teaching materials and the dissemination of good practices through teacher education" (Levrini, 2014, 178). These problems have been faced with different attempts of building different curricula from the classic ones, like in Guisasola (2009) or Miller (2010), where students can connect and experience in a deeper way what is meant by the theory, which are the implications from an epistemological point of view and what is the historic path that the theory followed during the years.

The problems related with the design of teaching materials are analysed in the "In/Pre-Service Teachers Formation". The works from De Ambrosis and Levrini (2010), de Hosson, Kermen and Parizot (2010) and Selcuk (2017) showed how both in-service teachers and pre-service teachers have difficulties with STR concepts and have difficulties in teaching those same concepts, since textbooks are usually made in such a way that do not allow them to follow their path. De Ambrosis and Levrini stressed how the appropriation path is paramount in the development of a good teaching program.

STR is an important step that can bridge the classical world physics and the modern one, in that the concepts presented in the theory can require also basic mathematical knowledge, within the reach of students attending high schools or the first years of modern physics courses at the university level.

STR is considered a paramount topic in helping students develop conceptual change, and consequently helps teachers develop materials and theories aimed at fostering this change.

Nonetheless, the connection between mathematics and physics in STR is a very little analysed topic, both in the PER field and the MER field. The works of Tzanakis (1999; 2000; 2002; 2016) and Galili (2018) focused on this relation and have shown how STR can be an excellent historical example with which to demonstrate the deep connection that exists between disciplines.

2.5.1 Considerations and Research Questions

What is evident from this review is that interdisciplinarity is not a widely explored topic . Moreover, LT are studied usually as an instrument that helps making demonstrations in order to obtain Relativistic Effects, but what is missing is an analysis of the different roles that LT can have in the theory and also the important role that these perspectives have in understanding the theory.

In order to address these open problems, the Research Questions that have guided the studies reported in the Third Chapter are:

1. Which are the different perspectives taken by the original authors of STR theory?
2. What are the roles that LT can have in understanding the theory?
3. What interdisciplinary approach can be implemented to realize activities aimed to foster deep learning and grasping the different roles undertaken by the two different disciplines in exploiting the cultural, cognitive, and epistemological value of the theory?

To deal with these questions, an interdisciplinary analysis has been made in the third chapter using the interdisciplinary framework of "Boundary Crossing and Boundary Object" from Akkerman and Bakker (2011) on the original papers of the theory from Lorentz (1904), Einstein (1905), Poincaré (1906) and Minkowski (2008). Based on this analysis are the activities described in the fifth chapter.

Chapter 3 - STR Original Papers’ Interdisciplinary analysis

3.1 The historical panorama and the foundation crisis of late XIX century and early XX century

"This is an age of critical self-questioning in physics. Physicists are now desperately trying to come to terms with many of the concepts, which were accepted without question in the 19th century when classical physics dazzled everyone with its brilliant and spectacular triumphs."

(Bose, 1996, p. 101)

"The hypothesis that space is not homoloidal (i.e. flat), and again, that its geometrical character may change with the time, may or may not be destined to play a great part in the physics of the future; yet we cannot refuse to consider as possible explanations of physical phenomena, because they may be opposed to the popular dogmatic belief in the universality of certain geometrical axioms – a belief which has arisen from centuries of indiscriminating worship of the genius of Euclid."

(Clifford, 1886, p. 226)

These words of Bose and Clifford fully encapsulate the sentiment present in the scientific world at the turn of the XX century, even if the situation was much more complicated than this. This feeling was due to a series of discoveries made in the nineteenth century in the scientific world in fields like physics, mathematics, or chemistry. These discoveries changed forever the way in which reality was understood, but the emergence of this feeling followed a very slow and irregular path.

Without following what happened step by step, we can recollect all the main aspects that led to the need of reconsidering fundamental concepts like space and time.

After the years where Newtonian physics was considered the highest point that would ever be touched in the description of mechanical phenomena, even with various critiques against the concept of force or the instantaneous action of gravitation, in the nineteenth century this feeling started to fade away. The discovery of the fundamental components of matter like the electron started to raise concerns about the study of their dynamics in the atoms and of molecules in general and raised the question if Newtonian mechanics was capable of explaining these kinds of motions. In that period Newtonian mechanics was considered at the same level of Euclidean geometry, i.e., a theory based on a few foundational principles having the same kind of definiteness as the axioms of Euclidean geometry.

Theories such as thermodynamics and electrodynamics then began to drop this type of reasoning based on the accuracy of the data and knowledge of what was happening in the system moment by moment. Due to the impossibility of following the development of systems with enormous numbers of molecules, scientists like Boltzmann changed the approach towards reality inserting statistics as a way to analyse nature. In this way, scientists started to think of new kinds of theories able to explain these behaviours.

One attempt made to explain the light behaviour in empty space was the hypothesis for which the whole universe was filled with some continuous substance called ether, even if the concept of ether was present already in Newton's time. As Bose said in his paper mentioned before, "atoms or subatomic particles are then floating in this ocean of ether. Light is nothing but the waves in this ocean" (Bose, 1996, p. 97). Ether therefore was considered a substance present everywhere that allowed transmissions of all kinds of waves, from radio waves to X-rays. Scientists in modern physics then tried to understand all phenomena occurring in the natural world in terms of the interactions amongst the particles inside the atom and their interactions with the waves in the ether.

A lot of experiments were made in this field trying to demonstrate the existence of ether. The most famous without doubt is the Michelson and Morley one, made many times since 1880 with the goal of demonstrating the change in the velocity of light with respect to the ether using an interferometer. The negative results of these experiments led to the birth of new attempts to explain the results, like Lorentz' one that will be treated in the next sections.

For reasons like these, in the end of the nineteenth century Newtonian physics was not as unquestionable as it was in the starting of that same century, also because at the turn of the century the common thinking the cultural configuration was changing from mechanical to electromagnetic and thermodynamical (Kragh, 1999).

But in the same period another unquestionable theory was starting to lose his primary role, i.e., Euclidean geometry.

Euclidean geometry is one of the oldest theories in the history of science and in the early years of the nineteenth century mathematicians were convinced that it was the correct idealization of properties of physical space and of figures in that space. The reigning thought in those years was still tied to the Kantian perspective, according to which certain principles about space are prior to experience and so the nature of the external world is known to us only in the way our minds oblige us to interpret it. On this ground, for him and his contemporaries, the physical world had to be Euclidean. The common conclusion was the uniqueness and necessity of Euclidean geometry (Kline, 1972). But as Riemann pointed out in his 1854 work, "geometry presupposes the concept of space, as well as assuming the basic principles for construction in space. It gives only normal definitions of these things, while their essential specifications appear in the form of axioms. The relationship between these presuppositions [the concept of space, and the basic properties of space] is left in the dark; we do not see whether, or to what extent, any connection between them is necessary, or a priori whether any connection between them is even possible" (Riemann, 1854, p. 135). The Euclidean geometry for him is not logically necessary, but a hypothesis based on empirical certainty (Riemann, 1854).

Moreover, one of the reasons why Euclidean geometry was not considered necessary from a logical point of view was the problem of the self-evidence of the fifth postulate that occupied mathematicians' minds since Euclid himself. This postulate (i.e., the parallel axiom, according to two straight lines intersected by a third straight line will meet on that side of the transversal where the sum of the interior angles is less than two right angles), was different from the others in that "no one really doubted his truth and yet it lacked the compelling quality of the other axioms" (Kline, 1972, p. 863).

A lot of substitute axioms were suggested but all of them had the same problem, since they were proved to be logically equivalent to Euclid's one. The simplest of those was enunciated by John Playfair introduced in 1795, that is the version used in modern books:

"Through a given point P not on a line l, there is only one line in the plane of P and l which does not meet l".

Therefore, the history of non-Euclidean Geometry began with the efforts to eliminate the doubts about Euclid's parallel axiom. What was made by many mathematicians at the same time, like Gauss, Lobachevsky, and Bolyai was trying to demonstrate that a logical geometry developed adopting a contradictory statement towards the fifth postulate was possible, and that "Euclidean geometry is not the only geometry that describes the properties of physical space to within the accuracy for which experience can vouch" (Kline, 1972, p. 869).

The impact non-Euclidean geometry had on both the mathematical and the physical world was tremendous, in that this theory proved that Euclidean geometry, based on axioms that were thought as self-evident truth, was not the only possible geometry adapt to describe the physical space.

The deep connection between geometries and the resulting view of the physical world is well expressed in the Clifford sentence quoted in the start of this section. In his work edited posthumous in 1886 he presents three different possibilities concerning the variation in the curvature of space:

- Space has a curvature varying from point to point that is not appreciable;
- Space is the same everywhere but its degree of curvature changes with the time;
- Space has everywhere a nearly uniform curvature, but this curvature has small variations in space and time.

These different visions have consequences on our experience in the physical world, in that changes in physical conditions may be due to changes in the curvature of space (Clifford, 1886).

Using Kline words,

"The problem of the parallel axiom was not only a genuine physical problem but as fundamental a physical problem as there can be" (Kline, 1972, pp. 880-881).

Apart from non-Euclidean geometries, other kinds of geometries were being born in that period, like the projective one. In order to recollect all these different research fields, Felix Klein in 1872 with the Erlangen program overcame the plurality of geometries in a higher-order perspective. Within Klein's Program, any geometry is the study of concepts, properties, and relations invariant under the transformations of a group. Unlike the great rapture generated by non-Euclidean geometries, Klein's work did not find opposition and did not overturn previous theories. Despite that, his new approach changed in a permanent way the mathematical world, because since then every geometric research has to specify in which geometry it is located and is a standard for revolutionary movements to elicit new classes of problems (Speranza, 1997).

3.2 Introduction to the original papers

In this section a presentation of the original texts will be made, with an explanation of why those papers are still relevant today and fundamental for the revolution mentioned in the previous section.

An important point that needs to be emphasized is that even if these papers have been studied for a long time now from many different points of view, they still have the great potential to bring out an interdisciplinary approach aimed to investigate the intertwining between mathematics and physics.

3.2.1 Electromagnetic phenomena in a system moving with any velocity smaller than that of light

This paper, published in 1904 in the proceedings of the Royal Netherlands Academy of Art and Science (KNAW), is subdivided into 13 sections.

I. The starting point of the paper tackles directly the main problem of Lorentz' work, i.e., "determining the influence exerted on electric and optical phenomena by a translation".
In his opinion, the solution for this problem is that "dimensions of solid bodies are slightly altered by their motion through the ether". To support this hypothesis Lorentz brings to the attention of the reader two experiments; the Michelson one and the Rayleigh and Brace, both with negative results. After this he talks about another experiment, made by Trouton and Noble, with the goal of detecting a turning couple acting on a charged condenser, whose plates make a certain angle with the direction of translation.

II. In the second section Lorentz says that these negative results are not the only reason why "a new examination of the problems connected with the motion of the Earth is desirable".
In his words, he is not happy with Poincaré's approach of inventing special hypothesis for each new experimental result:
"Poincaré has objected to the existing theory of electric and optical phenomena in moving bodies that, in order to explain Michelson's negative result, the introduction of a new hypothesis has been required, and that the same necessity may occur each time new facts will be brought to light. Surely this course of inventing special hypotheses for each new experimental result is somewhat artificial. It would be more satisfactory, if it were possible to show, by means of certain fundamental assumptions, and without neglecting terms of one order of magnitude or another, that many electromagnetic actions are entirely independent of the motion of the system." (Lorentz, 1904, pp. 12-13)

III. In the third section he starts from the fundamental equations of the theory of electrons (i.e., Maxwell Equations) and then, supposing that the system as a whole moves in the direction of x with a constant velocity, he develops those equations for this particular system.

IV. Then he starts to transforms those equations by a change of variables: $\frac{c^2}{c^2-w^2} = k^2$ and also $x' = klx, y' = ly, z' = lz, t'\left(\frac{l}{k}\right)t - kl\left(\frac{w}{c^2}\right)x$ (where t' is the local time). After defining two new vectors and explaining the meaning of l, he substitutes these new variables into the equations.

V. With his words, "these new equations lead to the conclusion that the new vectors may be represented by means of a scalar potential and a vector potential". Then, in this section as in sections 6 and 7 he proceeds with very precise calculations about forces, potentials and change of variables between different electrostatic systems in relative motion with respect to each other. The final result is an equation that describes the field produced by the polarized particle of the system.

VI. \

VII. \

VIII. At this point, Lorentz introduces an assumption; that is “that the electrons, which I take to be spheres of radius R in the state of rest, have their dimensions changed by the effect of a translation, the dimensions in the direction of motion becoming kl times and those in perpendicular directions l times smaller”. Basically, as he explains later, what he is saying it is that “all electrons in a moving electrostatic system moving with a velocity w are flattened ellipsoids with their smaller axes in the direction of motion”. After this, there is another assumption; that is “that the forces between uncharged particles, as well as those between such particles and electrons, are influenced by a translation in quite the same way as the electric forces in an electrostatic system”. The final results of these assumptions are that the translation from one system to the other will produce the deformation.

IX. In the ninth section Lorentz calculates the electromagnetic momentum of a single electron, but he enters into technicalities not relevant to our discussion.

X. In the tenth section Lorentz examines the influence of the Earth’s motion on optical phenomena in a system of transparent bodies. After some calculations he makes another assumption, that is “that the influence of a translation on the dimensions (of the separate electrons and of a ponderable body as a whole) is confined to those that have the direction of the motion, these becoming k times smaller than they are in the state of rest”. With this assumption he arrives at this result: “if, in the system without translation, there is a state of motion in which, at a definite place, the components of p,d,h are certain functions of the time, then the same system after it has been put in motion (and thereby deformed) can be the seat of a state of motion in which, at the corresponding place, the components p’,d’,h’ are the same functions of the local time.

XI. In this recap section he shows that his new theory is able to account for a large number of facts, like experiments on interference and diffraction. Nonetheless he stresses the point that his theory still has some reserves because it leads to some consequences that cannot be put to the test of experiments.

XII. Here Lorentz focuses on molecular motion and says that bodies in which molecular motion has a sensible influence or even predominates undergo the same deformation as the systems of particles of constant relative position.

XIII. In the thirteenth section there is a discussion about two series of measurements made by Kaufmann in 1902.

3.2.2 Sur la dynamique de l’électron (On the dynamic of the electron)

This paper is subdivided in two main parts: an introduction, more discursive, that briefly summarizes all the paper’s content, and the actual paper, very specific and written mainly in mathematical language, where Poincaré resumes all the contents treated in Lorentz’s paper correcting and motivating the reason of some different choices.

The paper’s introduction starts with a collection of negative experimental results regarding the determination of the absolute motion of the Earth. From here Poincaré introduces Lorentz and FitzGerald explanation, that is “the hypothesis of a contraction experienced by all bodies in the direction of motion of Earth and proportional to the square of the aberration: this contraction, which we will call Lorentz contraction, took into account the Michelson experiment and all those which had been done until now”, but also says that “this hypothesis would become insufficient if the relativity postulate were to be accepted in its full generality”. (Poincaré, 1906, p. 45)

In Poincaré's opinion what Lorentz succeeded in doing in his 1904 article was "to supplement it and amend it so as to bring it into full agreement with the Relativity postulate". (Poincaré, 1906, p. 46)

After this he explains Lorentz's idea and Langevin's proposal to modify it, but put emphasis on the fact that this theory does not respect the relativity postulate, and that is why he thinks that "it's necessary to go back to Lorentz's theory; but to keep it and avoid intolerable contradictions a special force has to be assumed which explains both the contraction and the two constant axes". (Poincaré, 1906, p. 46)

In determining this force, Poincaré found that "it could be compared to a constant external pressure acting on the deformable and compressible electron and its work is proportional to the variations in the volume of this electron". To prove this, under the hypothesis that the inertia of matter was exclusively of electromagnetic origin and that all the forces are of electromagnetic origin, he uses a "very simple calculation based on the principle of least action". (Poincaré, 1906, p. 46)

Afterwards Poincaré quotes Lorentz hypothesis that "the postulate is still true when there are forces other than electromagnetic forces", so he asks himself "what modifications it would force us to make to the laws of gravitation" and then starts to analyse the problem of gravitation, and in particular "if the propagation of gravitation is not instantaneous but occurs at the speed of light". (Poincaré, 1906, p. 46)

The topics covered in the second part are basically the same but treated in a deeper way. They are:

1. Lorentz transformation
2. Principle of Least Action
3. Lorentz Transformation and the Principle of Least Action
4. The Lorentz Group
5. Langevin Waves
6. Contraction of Electrons
7. Quasi-Stationary Motion
8. Arbitrary Motion
9. Hypotheses on Gravitation

3.2.3 On the Electrodynamics of Moving Bodies

The paper is subdivided in:

- Introduction;
- Kinematical part;
- Electrodynamical part.

In the introduction of "On the Electrodynamics of Moving Bodies", Einstein starts his reasoning from the "asymmetries resulting from Maxwell's electrodynamics applied to moving bodies" and from the "unsuccessful attempts to discover any motion of the earth relatively to the light medium". (Einstein, 1905, p. 37). Here he immediately makes it clear which are the two main revolutions underlying his work: raising the conjecture according to which "the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good" to the status of a postulate (later called "Principle of Relativity") and introducing another postulate, i.e. "that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body" (Einstein, 1905, p. 37-38). These two principles then would lead to a new kinematics that solves the contradictions of the electrodynamics of moving bodies.

After the introduction, the paper is divided into two parts: a Kinematical part and an Electrodynamical part.

In the Kinematical part we find 5 sections:

1. Definition of Simultaneity
2. On the Relativity of Lengths and Times
3. Theory of the Transformation of Co-ordinates and Times from a Stationary System to another System in Uniform Motion of Translation Relatively to the Former
4. Physical Meaning of the Equations obtained in Respect to Moving Rigid Bodies and Moving Clocks
5. The Composition of Velocities

This part is focused on the definitions underlying the theory and the consequences of postulates on measurement activities and transformations between systems in motion.

The first section is devoted to the definition of simultaneity in an operative way. This definition is made with the goal of having propagation of light isotropic in the given frame. Then he assumes that the ratio between the double of the distance between two clocks and the time needed for the light to go back and forth is equal to the universal constant c.

In the second section Einstein focuses on the principles, the "Relativity Principle" and the "Principle of the constancy of the velocity of light". Here he shows that simultaneity is relative because it depends on the reference frame from which the synchronized events are seen.

In the third section he derives Lorentz Transformations (LT) requiring that c stays the same in two different reference systems, one moving with respect to the other.

In the fourth section Einstein focuses on the physical meaning of the equations found in the previous section, using LT to obtain equations for length contraction and time dilation.

In the fifth section he shows how the composition of velocities is modified by the consequences of the two postulates.

In the Electrodynamical part we also find 5 sections:

6. Transformation of the Maxwell-Hertz Equations for Empty Space. On the Nature of the Electromotive Forces Occurring in Magnetic Field During Motion
7. Theory of Doppler's Principle and of Aberration
8. Transformation of the Energy of Light Rays. Theory of the Pressure of radiation Exerted on Perfect Reflectors
9. Transformation of the Maxwell-Hertz Equations when Convection-Currents are Taken into Account
10. Dynamics of the Slowly Accelerated Electron

In this second part Einstein concentrates on the results of the postulates on the Electrodynamical theory of Maxwell-Hertz and focuses on the application of the new kinematics to electrodynamics.

In the sixth section Einstein proved the covariance of the homogeneous Maxwell-Lorentz equations and used the relevant field transformations to remove the theoretical asymmetry presented in the beginning of the paper.

In the seventh section he uses the transformations of a plane monochromatic wave to derive the Doppler effect and stellar aberration.

In the eighth Einstein derives the transformation law for the energy of a light pulse.

In the ninth he obtains the covariance of the inhomogeneous Maxwell-Lorentz equations.

In the tenth section he obtains the relativistic equation of motion of an electron in an electromagnetic field.

3.2.4 Space and time

In this paper, that is a transcription from a lecture given at the 80th Meeting of the Natural Scientists in Cologne on September 21, 1908, Minkowski starts with a simple and yet very strong sentence, that is the very famous “from now onwards space by itself and time by itself will recede completely to become mere shadows and only a type of union of the two still stand independently on its own”.

After this brief introduction, we find 5 sections:

1. In the first one, Minkowski explains “how to move from the currently adopted mechanics through purely mathematical reasoning to modified ideas about space and time” (Minkowski, 1908, p. 39). To do this end he starts from a description of Newtonian physics as a system made by 2 main groups of transformations. In an attempt to merge these two groups, he introduces his new approach where x,y,z,t are the 4 coordinates of the space-time. He also introduces new concepts like worldpoint and wordline. For this introduction he uses a graphic approach and starting from this graph he builds the rest of his new way of analysing the concept of event. [Insert graph here]. Also, in this section he introduces his metric, very important for the development of the concept of space and time as a whole. In this metric we can find the positive parameter c, identified with the velocity of the propagation of light in empty space. To end the section, Minkowski introduces a particular group of transformations called G_C (G_∞ in the limit c=∞) that allows to switch between coordinates x, y, z, t and x’, y’, z’ and t’ and states that, in order to satisfy the invariance of the laws of nature with respect to this group, “one can still change the reference system according to the transformations of the above group G_C arbitrarily without changing the expression of the laws of nature in the process” (Minkowski, 1908, p. 42).
2. In the second section, Minkowski introduces an axiom, namely: “with appropriate setting of space and time the substance existing at any worldpoint can always be regarded as being at rest” (Minkowski, 1908, p. 43). After doing this he explains which are the reasons that forced him to the changed view of space and time and if this new vision provides advantages for the description of the phenomena. For him, “the impulse and true motivation for accepting the group G_C came from noticing that the differential equation for the propagation of light waves in the empty space possesses that group G_C” (Minkowski, 1908, p. 43). From this point, he analyses Lorentz’ hypothesis according to which, everybody moving at a velocity v must experience a reduction in the direction of its motion and shows that “the Lorentzian hypothesis is completely equivalent to the new concept of space and time” (Minkowski, 1908, p. 44). In the end of the section he discusses the work of Einstein, “who first realized clearly that the time of one of the electrons is as good as that of the other” but says that “the word relativity principle used for the requirement of invariance under the group G_C is very feeble since the meaning of the postulate is that through the phenomena only the four-dimensional world in space and time is given, but the projection in space and time

can still be made with certain freedom" and for this reason he prefers the term "postulate of the absolute world", or world postulate (Minkowski, 1908, p. 43).

3. In the third section he introduces the light cone identified by worldlines and an arbitrary worldpoint as the origin of space-time. After this he distinguishes between timelike and spacelike vectors and then introduces the proper time of the substantial point. In the last part of this section, using the first and second derivatives of x,y,z,t with respect to the proper time τ he introduces the velocity and the acceleration vectors through a curvature hyperbola, on which he demonstrates some particular relations between those vectors.
4. In the fourth section Minkowski demonstrates that "the adoption of the group G_C for the laws of physics never leads to a contradiction" (Minkowski, 1908, p. 49). To realize this, he expresses the need for a revision of all physics based on the assumption of this group and in particular he focuses on the concept of force. In doing this revision he also treats the conservation of energy.
5. In the last section, he focuses on the advantages resulting from the world postulate in describing the effects from an arbitrarily moving point charge according to the Maxwell-Lorentz theory and the ponderomotive action of an arbitrarily moving point charge exerted on another arbitrarily moving point charge. In doing this, he shows that "according to the world postulate, the disturbing disharmony between Newtonian mechanics and the modern electrodynamics disappears by itself" (Minkowski, 1908, p. 52). After this, always regarding the world postulate he analyses the status of the Newtonian law of attraction. As a result, he shows that "the proposed law of attraction associated with the new mechanics is no less well suited to explain the astronomical observations than the Newtonian law of attraction associated with the Newtonian mechanics" (Minkowski, 1908, p. 53). In the end of the lecture, he states that "the validity without exception of the world postulate is the true core of an electromagnetic world view which, as Lorentz found it and Einstein further unveiled it, lies downright and completely exposed before us as clear as daylight".

The last sentence of this paper encloses all Minkowski's thought:

"With the development of the mathematical consequences of this postulate, sufficient findings of its experimental validity will be arrived at so that even those to whom it seems unsympathetic or painful to abandon the prevailing views become reconciled through the thought of a pre-stabilized harmony between mathematics and physics." (Minkowski, 1908, p. 53)

3.2.5 The Genesis of the Theory of Relativity

At this point it is useful to take stock of the situation, and to do this we will refer to the article "The Genesis of the Theory of Relativity" by Darrigol (Darrigol, 2006). In his work, Darrigol traces the history of the Special Theory of Relativity (STR) from Maxwell's equations to Einstein 1905 paper "On the Electrodynamics of Moving Bodies".

Talking about Lorentz role in the development of the theory, Darrigol stresses that for him the transformed coordinates and fields were only mathematical aids with no direct physical meaning, but he used those only with the goal of resolving problematic differential equations. Also, the local time t' in his work has not the same meaning given to it by Einstein or Poincaré but is just a convenient choice of coordinates depending on the abscissa. For him, the true physical quantities were only the fields representing the states of the ether and the absolute time t.

Differently from Lorentz, Poincaré provided a physical interpretation for the transformed fields and also for the transformed time t'. With this interpretation therefore was possible for him to elaborate LT from a new perspective; that is, having the invariance of optical phenomena as a direct consequence of the formal invariance of the Maxwell-Lorentz equations. Despite this, in Darrigol's opinion Poincaré was not able to redefine the concepts of space and time, because the quantities transformed with the LT were only apparent states, while the true states were still those defined with respect to the ether. Moreover, Poincaré never accepted the rationale underlying Einstein's work on SRT until his death in 1912.

In Einstein's work instead Darrigol put emphasis on the fact that in his view and in his understanding of the relativity principle there was no ether and no absolute frames, but all inertial frames were equivalent. Differently from Lorentz and Poincaré, for him time and space coordinates defined in these frames were equivalent. Also in all these frames, due to the Relativity Principle, the constancy of the velocity of light had to hold without using the ether but postulating it separately. To obtain all of these without falling into contradictions, Einstein understood that he had to redefine time according to the light postulate, and in this way LT were implied without having to resort to Maxwell-Lorentz equations.

In order to recapitulate the process made by these three authors and by all of the others that took part in the development of the theory, Darrigol says that

"Most of the components of Einstein's paper appeared in others' anterior works on electrodynamics of moving bodies. Poincaré and Alfred Bucherer had the relativity principle. Lorentz and Larmor had most of the Lorentz transformations, Poincaré had them all. Cohn and Bucherer rejected the ether. Poincaré, Cohn, and Abraham had a physical interpretation of Lorentz's local time. Larmor and Cohn alluded to the dilation of time. Lorentz and Poincaré had the relativistic dynamics of the electron. None of these authors, however, dared to reform the concepts of space and time. None of them imagined a new kinematics based on two postulates. None of them derived the Lorentz transformations on this basis. None of them fully understood the physical implications of these transformations. It all was Einstein's unique feat." (Darrigol, 2006, p. 25)

3.2.6 The importance of studying original papers

After this introduction, we focus on the reasons why analysing these texts is still important today. Despite being one hundred years old and more, these papers represent the birth of one of the most important theories in modern physics, and still today there are a lot of studies, which use these papers as a starting point to carry out historical and epistemological analyses, but also to introduce the theory of relativity using directly the words of its founding fathers (Levrini, 1999, 2002, 2014; Tzanakis 1999, 2000, 2002, 2016; Villani & Arruda, 1998; Messager et al., 2012; Provost & Bracco, 2016).

Moreover, these texts have already been used from a didactic perspective. In the work done in this thesis an analysis will be conducted on these original papers by using some particular interdisciplinary lenses introduced by Akkerman and Bakker and called "Boundary Crossing and Boundary Objects". This analysis has the goal to demonstrate that some objects peculiar to this theory (LT) can be seen in different ways based on the role given to them and can activate different ways of thinking about the theory itself. In particular, when a theory is at the boundary between disciplines, like the Special Theory of Relativity is at the boundary between physics and mathematics, it is always difficult to understand what is the role that some objects can play in it. Trying to explore the various meanings given to those objects by the founders of the theory is fundamental, as in this way we can understand what vision of the theory they had in mind.

3.3 Interdisciplinary framework

After this introduction, it is necessary to introduce the interdisciplinary lenses of Akkerman and Bakker (2011) as they are intended by the IDENTITIES project. An in-depth analysis will be made on the framework developed by these two articles, explaining in particular why this framework can be used in the Science Education field.

3.3.1 Boundary crossing and boundary object

The framework of "Boundary crossing and boundary objects", introduced by Akkerman and Bakker in 2011, "offers an understanding of boundaries as dialogical phenomena". In this framework, used inside the IDENTITIES project, they reviewed many different papers concerning interdisciplinarity and revealed the presence of "four potential learning mechanisms that can take place at boundaries: identification, coordination, reflection and transformation" (Akkerman & Bakker, 2011). These mechanisms show how sociocultural differences and discontinuities between various fields can interact to create new resources for the development of practices inside those fields.

In order to understand totally their approach is important to understand what they mean for boundaries. For them,

"A boundary can be seen as a socio-cultural difference leading to discontinuity in action or interaction. Boundaries simultaneously suggest a sameness and continuity in the sense that within discontinuity two or more sites are relevant to one another in a particular way. [...] Boundaries belong to both one world and another, however they also reflect a nobody's land, belonging to neither one nor the other world ".

(Akkerman & Bakker, 2011, p. 133)

To describe the different modalities of interaction between boundaries two concepts are fundamental in their work: boundary object and boundary crossing.

Boundary crossing expresses the way in which professionals usually interact between them when they work in fields unfamiliar for them and therefore, they are forced to collaborate in order to achieve a common goal. In doing so, they use different methodologies and practices that come from their respective field.

During the process of crossing there is the possibility to use the same object that can be seen as an artifact that fulfils a specific function in bridging intersecting practices (Star & Griesemer, 1989, p. 393). These objects are called boundary object, i.e.,

"Those objects that both inhabit several intersecting worlds and satisfy the informal requirements of each of them… [They are] both plastic enough to adapt to local needs and the constraints of the several parties employing them, yet robust enough to maintain a common identity across sites. They are weakly structured in common use, and become strongly structured in individual site use".

(Star & Griesemer, 1989, p. 393)

An interesting perspective about boundary objects is the one from William and Wake. For them boundary objects are something that can be invisible or taken-for-granted mediations across sites but if analysed carefully they may provide learning opportunities (Williams & Wake, 2007).

After an in-depth discussion about what are the common perceptions of boundaries in the literature, Akkerman and Bakker describe a mechanism that constitutes to the learning potential of Boundary Crossing, i.e., the four mechanisms of learning at the boundary.

- The first mechanism is IDENTIFICATION, which is about comparing the differences between practices between them. The boundary crossing here is seen as a process in which the line between the two disciplines is not clear-cut for different reasons, like instability or overlapping between practices. This comparison leads to questions about the identities of the various sites participating in the exchange and to a renewed vision of the sites themselves and their respective practices.
  Akkerman and Bakker then describe two common processes of identification described in the studies, that are othering and legitimating coexistence.
  "The learning potential resides in a renewed sense making of different practices and related identities".

- The second mechanism is COORDINATION, which consists of creating cooperative and routinized exchanges between practices. Movement and dialogue between practices are encouraged in order to keep the flow of work between all participants in the joint work. Here the potential of this mechanism stands in overcoming the boundary, not in reconstructing it.
  Four processes of coordination are discussed, that are communicative connection, efforts of translation, enhancing boundary permeability and routinization. These processes use common instrumentalities, i.e., boundary objects, to connect and coordinate the different sites involved in the process.

- The third mechanism is REFLECTION, which is about expanding one's perspectives on the practices.
  In this process the boundary crossing process consists in understanding the differences between practices and in discovering more about the practices involved. The Reflection mechanism involves perspective making and perspective taking. Through this mechanism people can look into the world in an enriched way and in doing so they can enrich the various identities involved in the process.

- The last mechanism is TRANSFORMATION, which is about collaboration and codevelopment of (new) practices. "Transformation leads to profound changes in practices, potentially even the creation of a new, in-between practice, sometimes called a boundary practice."
  The essential traits of the transformation process are confrontation, recognizing a shared problem space, hybridization, crystallization, maintaining uniqueness of the intersecting practices and continuous joint work at the boundary.

In order to recap all of these categories, the authors made some points about the question of how these mechanisms relate to one another:

- "Identification is about constructing and reconstructing boundaries, whereas the other mechanisms are more about transcending boundaries.
- Identification and reflection mechanisms mainly reflect meaning-oriented learning processes, whereas both coordination and transformation reflect more practice-based learning processes.

- The coordination mechanism seems opposite to transformation, as the former reflects a smooth, effortless, and routine process of people or objects moving back and forth between practices, whereas the latter involves confrontations and continuous joint work.
- Identification and reflection, both involving the explication and visibility of perspectives, seem conditional for transformation because in the latter boundaries need to be encountered and contested before being put to use for co-developing practices".

As a conclusion, the authors say that "dialogical engagement at the boundary does not mean a fusion of the intersecting social worlds or a dissolving of the boundary. Hence, boundary crossing should not be seen as a process of moving from initial diversity and multiplicity to homogeneity and unity but rather as a process of establishing continuity in a situation of sociocultural difference. This holds also for the transformation mechanism, in which something new is generated in the interchange of the existing practices, precisely by virtue of their differences".

3.4 Interdisciplinary analysis

In this section an interdisciplinary analysis of the STR original papers from Lorentz, Poincaré, Einstein and Minkowski is presented using the interdisciplinary lenses of boundary crossing and boundary objects presented in the previous section. To do so, I have chosen some excerpts from these papers that in my opinion clearly show what was the epistemological approach behind their works. They are discussed in the next sections and are quoted in full in the Appendix.

Of course, there is not a clear demarcation between mathematics and physics in these excerpts. Also because of that, it is really helpful to apply the interdisciplinary lenses of Akkerman and Bakker to them in order to point out what image of interdisciplinarity comes out from these texts.

The excerpts chosen for this analysis contain fundamental passages for the development of STR, from the first definition of "Lorentz transformations" as they were called by Poincaré, to the definition of simultaneity from Einstein, and end up with Minkowski's contribution to move away from considering the concepts of space and time as two separate entities, up to the birth of the concept of space-time.

3.4.1 Lorentz 1904 Analysis

From the Lorentz paper, three excerpts appeared particularly relevant for the analysis. They are from sections 2, 4, and 8, and are reported in the Appendix.

The reasons why Lorentz finds appealing and useful to examine problems related to the Earth's motion (i.e., problems related to the ether) are two: first, to explain the experimental results obtained by Michelson and Morley, and second, to demonstrate with "certain fundamental assumptions that many electromagnetic actions are entirely independent of the motion of the system". Here we notice that the starting point of Lorentz reasoning is anchored in the world of experience, i.e., it belongs fully to the realm of physics.

From here, the next step taken by Lorentz brings the discourse into the world of mathematics. In fact, in the second excerpt we have selected, he makes a particular choice of variables (that is the introduction of parameters k and l) and, thanks to that, he finds a particular set of transformations. These transformations, as Darrigol made clear, have no physical meaning for him, but are just "mathematical aids that helped him to resolve some particular differential equations".

In the third excerpt (that is section 8 in his paper) he comes back to the physics world, making two assumptions about the change of electron's dimensions in the direction of translation and about the influence of translation on forces between uncharged particles and between uncharged particles and electrons (assumed similar to that between charged particles in an electrostatic system). With the aid of these assumptions and the transformations, he is able to demonstrate that "a translation will produce a deformation", succeeding in motivating the results obtained by Michelson's experiments.

The relation between mathematics and physics in this work can be seen as a process of Coordination, as intended by Akkerman and Bakker. In fact, between the two disciplines there is "a communicative connection that is established by tools that belong to the different 'areas'". In particular, this tool can be seen in the transformations, that Poincaré, in his 1905 paper, called "Lorentz transformations".

LT assume in this paper the role of object that triggered the mechanism boundary crossing, that is the role of boundary object as intended by Star & Griesemer (1989).

Going deeper into what has just been said, LT have the peculiarity of belonging at the same time to two different worlds, the mathematical world, and the physical world, in each of which they have different meanings.

Lorentz motivated his choices with the goal of preserving Maxwell equations, and in doing so he made mathematical assumptions that at first glance appeared absurd, "extreme fantastical hypothesis" as it was called by Minkowski.

The image that comes out of this analysis is that of a theory not yet mature and too deeply rooted in the past to see the potential within itself.

3.4.2 Poincaré 1906 Analysis

Regarding Poincaré's work, the second part of the paper is very specific and written mainly in mathematical language. For our purposes, the first part of the introduction, where the author resumes all the work he has done, and on the first and fourth section, since they contain more information for an interdisciplinary analysis. They are reported in the Appendix.

In these excerpts, Poincaré, differently from Lorentz, never leaves the "physical world" to enter into the "mathematical" one, but in every step he develops a joint reasoning that belong to both the "realities". More specifically, he chooses the strategy to explain the physical reason for every mathematical transformation he does.

Recalling all the experimental proofs against the possibility of showing the absolute motion of the Earth, Poincaré elevates this impossibility to a law of nature. This law, called Relativity Postulate, is for him mandatory to comply with, and the reason why he considers Lorentz theory better than the others is that, after the 1904 review of his previous thoughts, Lorentz was the only one capable of keeping a compatibility with the Relativity Postulate. The crucial contribution made by Poincaré at this point is the attempt to "keep it [Lorentz theory] and avoid intolerable contradictions". To do so, the French scientist assumes a special force, that "could be compared to a constant external pressure acting on the deformable and compressible electron and its work is proportional to the variations in the volume of this electron".

The same line of thought can be seen in the first section (and in general in all his paper). Even if from this point onward he becomes very specific and mathematical in his demonstrations, he never loses his ties with the physical world.

The differences between the two works can be seen in the approach that the two scientists have towards the theory, the disciplines, and the experimental results. If Lorentz, as we have already shown, made some assumptions in the attempt to find a consistency with the experimental results, Poincaré demonstrates each step made by referring to a "'*physique des principes*' in which a few general principles served as guides in the formation of theories" (Darrigol, 2006). This trait is characteristic of Poincaré, who is considered a polymath and "the Last Universalist" (Ginoux & Gerini, 2013). As Darrigol noted, and as we have already said before, Poincaré "provided a physical interpretation of the transformed time t', and the transformed fields e' and b', which only referred to a fictitious system for Lorentz" (Darrigol, 2006).

Using our interdisciplinary lenses, we can describe this way of reasoning as a process of Transformation between the two disciplines. The dialogue between them is continuous, it is present in any part of the text. Reading Poincaré work we are able to analyse experimental results from a new perspective, every assumption is backed from a mathematical demonstration and in this way assumes new physical meaning. If this can be seen also in Lorentz work, here in the dialogue between the disciplines we are unable to distinguish the different voices. The final result is the merging of the two in a unique world, the "*physique des principes*", where there is no meaning in the distinction between the two disciplines.

Talking about the role of LT, on which Poincaré dedicates the third and fourth section, here we can see a sort of legitimation of them. Indeed, in the third section Poincaré demonstrates them using the Principle of Least Action and in the fourth emphasizes the fact that LT form a group.

Both of these two arguments are made with the intent of giving an interpretation from two different perspectives: in the third section Lorentz wants to attach physical meaning to them, in the fourth he proves that they are not only a combination of equations but actually, inside them, there is a more significant truth, that is the possibility of seeing them as a group, that is a finite or infinite set of elements together with a binary operation (called the group operation) that together satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property. This new vision, as we will see later, is the first step towards a mathematical construction of the theory that will be totally completed by Minkowski.

Therefore, to sum up the discussion, LT assume here a bridge-role between the disciplines. The metaphor of the bridge can be seen here as a two-way metaphor: from a certain point of view LT are that boundary object that allows the dialogue between the two disciplines and works as a foundation for the theory; at the same time each of the two pillars of this bridge (i.e., the two disciplines) takes a different shape thanks to the influence of the other. It is thanks to the demonstration by the Principle of Least Action (physical meaning) that the transformations have a base hard enough to be a group (mathematical meaning). But it is thanks to the fact that the transformations form a group (mathematical meaning) that we are able to construct a theory of space and time like the one made by Minkowski (physical meaning).

3.4.3 Einstein 1905 Analysis

As for Einstein's work, all the paper should be analysed for the "magnificent architecture of this memoir" (Darrigol, 2006, p. 23). However, for our purposes, I will focus on some excerpts taken from sections 1, 3 and 4 (see Appendix).

There are two important sentences in the first section, which enclose and show the relationship between mathematics and physics according to Einstein, as well as the differences between his approach and Lorentz one. The sentences are:

"Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by 'time'" (Einstein, 1905, p. 2).

"With the help of certain imaginary physical experiments, we have settled what is to be understood by synchronous stationary clocks located at different places, and have evidently obtained a definition of 'simultaneous', or 'synchronous', and of 'time'" (Einstein, 1905, p. 3).

With the first one Einstein clearly shows the order between the two disciplines. For him, a mathematical description of the phenomena has no meaning if there is not an explicit definition of what is meant by "time", i.e., is fundamental to start from reality, from the physical world, and then there is the possibility to describe it using mathematics.

The second sentence I have reported is important for his methodology. In fact, in order to define terms like “simultaneous” and “time”, he suggests “imaginary physical experiments”. Thought experiments are very frequent in his works, and are representative once again for his way of reasoning. Only starting from what is obvious in the real world he is able to “evidently obtain” the definitions he is looking for. There are no assumptions of any kind to explain experimental results (like Lorentz), nor mathematical demonstrations to prove the validity of what he has found (like Poincaré). Here everything is a logical consequence of the natural evidence raised to the level of postulate in the introduction of the paper.

Section 3 is about LT. To obtain these equations, Einstein starts from the description of a real situation, made by two moving systems of co-ordinates. After having defined the physical quantities involved in this situation, what Einstein does is “to find the system of equations connecting these quantities”. To accomplish this goal Einstein bases all his reasoning on the construction that he has made in the first section about the concept of simultaneity and the operational way to define it. The demonstration that follows is not based on assumptions, as Lorentz did, but on general physics evidence, i.e., the two postulates that he introduces in the introduction of this paper and the properties of homogeneity and isotropy of space and time.

What follows are only logical reasonings that come from those initial postulates, and in the end, he arrives at the same results of Lorentz and Poincaré.

In the fourth section Einstein discusses the effects of LT on bodies when observed from two different systems, one moving with respect to the other. The two main results, i.e., the contraction of lengths and the dilation of time, are obtained without using experimental results, but just following “imaginary physical experiments”, that are always based on the two postulates and on the equations he has obtained before.

At this point it is clear that, in this work, there is a specific order between mathematics and physics: for Einstein physics comes before everything, but more than physics what is really important for him is to respect those facts that are “evident” and ca ne operationally measured. Basing his reasoning on that, he is able to construct a solid theory that does not feel the need of experiments or assumptions to stand alone. The only experiment present in the paper, a thought one, is the one with which Einstein starts his introduction, that is “the reciprocal action of a magnet and a conductor” These kinds of experiments are an instrument used many times by him in his works. Differently from the others, Einstein did not explicitly mention the Michelson-Morley experiment and the paper includes no reference at all.

After having put these fundamental bases, and using only logical reasonings he finds the equations that agree with what he found previously. Analysing these logical reasonings, we are able to extrapolate his view of Mathematics. Einstein's approach toward Mathematics is unique, different from the ones of Poincaré and Lorentz.

Einstein is not interested in revolutionizing the mathematical world as Poincaré did with geometry and with the group approach from Felix Klein, nor uses mathematics as a pure instrument to explain experimental results. What he takes from Mathematics is the background reasoning, the logic that stays behind demonstrations and argumentations.

As he will write some years later,

“…pure mathematical construction enables us to discover the concepts and the laws […] which give us the key to the understanding of the phenomena of Nature. Experience can […] guide us in our choice of serviceable mathematical concepts [and] remains the sole criterion of the serviceability of a mathematical construction for physics, but the truly creative principle resides in mathematics” (Einstein, 1934, p.167).

Using that approach he is able to build a theory consistent and revolutionary, where every step taken is motivated by what comes before.

In this perspective, also LT assume a different role in the theory and can be seen under a different light. If for Lorentz they were a means to motivate experimental results, and for Poincaré they were a fundamental object with a physical meaning but also with a strong mathematical structure (i.e., they form a group), for Einstein the transformations are a result of the reasonings made following the postulates (based on physical-world evidence) with mathematics logic attitude. In his work the transformations are not the goal, they are not assumed, but they are a consequence of reality useful to describe it and by which it is possible to understand and finally resolve the asymmetries present in the previous theories.

Using the Boundary Crossing and Boundary Object framework, we can describe the interplay between mathematics and physics in Einstein paper predominantly as a mechanism of Reflection. This choice is due to the fact that Einstein applies the logical reasoning of mathematics to the physical world and, looking at the physics through these "eyes", he is able to build a theory that allows us to see the world in a different way. The sharing of methods between the different areas leads to a new understanding of the disciplines themselves: in particular, physics comes out from this interplay as a more solid branch of knowledge, strong of this new way of building a theory using a method of deduction that belongs to mathematical demonstrations. Looking through the eyes of mathematics we are able to see physics in a new viewpoint: therefore, the interplay between the different areas has as a result a new way of seeing things and understanding reality.

3.4.4 Minkowski 1908 analysis

Ultimately, regarding Minkowski's paper (Minkowski, 1908), it is interesting to focus on the first 2 sections (see Appendix). Also, in this case the excerpt is quite long, but analysing it all allows us to really understand his vision about the relation between mathematics and physics.

The first sentences of the first section are a manifesto for Minkowski's thought. Here in fact, he explains his goal, his method, and his view about the relation between physics and mathematics and the reason why he felt necessary to elaborate his new approach.

For him, the foundations of his "view of space and time" reside in "the domain of experimental physics" and this is in his opinion the real strength of this new view.

The method that he uses is presented in the third sentence of the first section, i.e. "I want to show first how to move from the currently adopted mechanics through purely mathematical reasoning to modified ideas about space and time". After this he considers the equations of Newtonian mechanics, focusing on their invariances. These two invariants, that are preserving their form "when subjecting the specified spatial coordinate system to any change of position" and "when any uniform translation is impressed upon it", represents a certain group of transformation, but as Minkowski said they were never studied together maybe for their "entirely heterogeneous character". The turning point of his theory comes now, because he says that it is the "composed complete group as a whole that gives us to think". From this point on, he proceeds in a geometrical demonstration using a graphic approach to explain why this new vision can unite space and time as a whole. In doing so, he introduces a different way to describe objects in space and their movement (i.e., worldpoint and worldline), and he does this starting from experimental evidence that motivates his choices and allows him to proceed with his mathematical demonstrations.

After having explained the geometrical representation of two systems (one moving with respect to the other) and having set the transformations between these two systems he arrives at the conclusion that these transformations constitute a group which depends on a parameter c, that is the velocity of propagation of light in empty space. This parameter is fundamental in the geometric construction of the description of the two systems because it allows him to "connect the orthogonality of space with the freedom of choice of the direction of the time axis". What comes from this construction is the hyperbola equation, that is the caliber with which he is able to switch from one set of coordinates to the other.

In the second section Minkowski focuses more on the effects that moving systems have on bodies and using his geometrical system he is able to demonstrate the Lorentz effect of length contraction. For doing that he introduces a fundamental axiom, that is: "With appropriate setting of space and time the substance existing at any worldpoint can always be regarded as being at rest". This choice allows him to demonstrate that any velocity is always smaller than c, that here finds his real meaning, that is the upper limit for all velocities. This parameter is critical for the building and the motivation for accepting the group of transformations GC, because "differential equation for the propagation of light waves in empty space possesses that group". From here Minkowski is then able to show that "Lorentz hypothesis is completely equivalent to the new concept of space and time, which makes it much easier to understand". Using the hyperbola as a caliber for switching between systems he succeeded in obtaining the same results obtained by Lorentz without new "extremely fantastical hypothesis", but just using a geometrical reasoning built on experimental evidence.

In the last part of the section, he discusses the influence that Einstein had on the concepts of space and time, but he says that neither Einstein nor Lorentz "shaked" the concept of space because "only the audacity of mathematical culture" can achieve that result. In fact, considering time as a variable with the same validity of space coordinates is a paradigm shift achievable only from a mathematical perspective. Using Minkowski's words, "the somewhat greater abstraction associated with the number 4 does not hurt the mathematician", where the number 4 stays for the 4 dimensions, that are x, y, z, and t.

Therefore, from this description is evident that for him the real world stays in the realm of physics, and mathematics is used in two different ways: from one point of view mathematics is used as an example of theory based on axioms, with a clear deductive logic that allows the construction of stand-alone demonstrations; from the other it allows him to describe the real world giving him instruments to connect different points in space and time. In his work, differently from the others already analysed, we find a new approach to obtain the transformation laws, the graphical-geometrical one, never used before him.

The geometrical description of reality introduced in this paper is fundamental for the development of physics, because after this moment, as Minkowski himself announces at the beginning of his paper, "space by itself and time by itself will recede completely to become mere shadows and only a type of union of the two will still stand independently on its own".

We can describe this process as a mechanism of Transformation because there is a never-ending dialogue between the two disciplines and every step is taken reasoning on both at the same time, leading to a change in how physics is considered and understood until then. Experimental physics gave the basis for the theory, mathematics worked as a fundamental support for the description of phenomena. The merging of these two gave birth to a new concept and a new view of the world, laying the foundations for the creation of the space-time concept.

3.5 Interdisciplinary Analysis results

In order to illustrate the results obtained by this analysis it is useful to make an overview of what has just been said.

In the four papers analysed in the previous section there are four different ways of seeing the relation between mathematics and physics.

- In the Lorentz paper there is a learning mechanism of Coordination between mathematics and physics, linked by LT that in this mechanism assume a boundary object role.
- In Poincaré work the learning mechanism acting between the two disciplines is a process of Transformation. The final result of his work is a merging of the two disciplines into one unique approach that can be called "*physique des principes*".
- The predominant learning mechanism present in Einstein work is the Reflection one. Applying the logical reasoning of mathematics to the physical world he is able to show from a different perspective the latter, changing how reality is understood.
- Eventually, in Minkowski's paper the learning mechanism in place is Transformation, in that the merging of geometry and physics gave birth to the concept of space-time and led to the foundation of what would have then been general relativity.

Talking about LT, the boundary object nature emerges strongly from this analysis. As shown in previous sections, LT can be seen under a different light in each of the four works considered, and in every work, they bring out a different vision of the theory. They are at the same time:

- a mathematical aid to prove experimental results;
- a set of equations demonstrated by the Least Action Principle;
- a group of transformations;
- the logical results of the Relativity Postulate and the constancy of c (as well as the assumption of the homogeneity and isotropy of space and time);
- the geometrical result that comes from changing the reference system using the hyperbola as a caliber.

Putting emphasis on these different visions allow to have a deeper comprehension of the theory in the sense that considering LT as a multifaceted object can be a great step towards the real understanding of how disciplines like mathematics and physics are deeply interconnected for a long time. Nonetheless Special Theory of Relativity is a clear example of how different aspects and methods of science can cooperate to build new knowledge.

Chapter 4 - Blended Learning

4.1 Introduction

The Blended Learning concept, or hybrid and mixed learning, was born around the end of the XX century, and it was applied in schools and universities from the beginning of the 2000.

The Blended Learning expression does not have a specific definition, but with this expression we refer to a way of teaching in schools that blends together a classic face-to-face learning in classrooms and an online learning: the online part can be administered in class or at distance with computers or other devices, like smartphones or tablets.

For Graham (2004) building a blended learning system means to combine standard face-to-face teaching with a computer-based teaching. On the same line we can find other definitions, coming from the same period (1990-2010), for which a learning/teaching process can be defined blended if it is realized both in class and online, with a proportion of the two parts that varies between 50:50 and 70:30.

Today, when we talk about Blended, we refer to a teaching/learning approach capable of reaching at the same time students in classrooms and students at distance: in this case the exact terminology is Blended Synchronous Learning (Bower et al, 2014).

In both cases, the goal of this kind of teaching/learning method it is the same: trying to update the scholastic teaching in order to make it more relevant and challenging for the new generations. At the same time, this kind of approach tries to put the student at the center of the learning process, aiming to develop competence-based learning rather than one based on the memorization of concepts.

Using online methods and devices can help both teachers and students in having available a great number of resources and instruments usable all day long next to the standard books.

In the last 15years a lot of studies claimed that blended learning improves the quality of teaching process on various levels, like: an upgrade of results obtained by students (López et al., 2016); a better understanding of studied arguments improving student satisfaction (So, 2009); reducing dropout rates (López-Perez et al., 2011). Chandra and Watters (2012), using a website that they developed using the cognitive apprenticeship framework, showed that for a group of nearly 50 students in Australia ‘the web-based learning experience benefitted the students in the treatment group. It not only impacted on the learning outcomes, but qualitative data from the students suggested that it had a positive impact on their attitudes towards studying physics in a blended environment’. Li and Tang (Li & Tang, 2017) showed that ‘blended learning is useful, powerful and realizable. Teaching physics with blended learning increases the teaching capacity, broadens the teaching “space”, and extends the teaching “time”, which allows instructors to easily teach and students to happily learn’.

Bazelais and Doleck (2018) applied the blended learning method in a college mechanics course to a group of 71 students. In their results they have found that ‘The findings suggest that the implementation of a well-designed blended classroom approach can be highly effective and can have a positive impact on both the quality of instruction and students’ learning outcomes. Importantly, students in the blended learning group appeared to experience greater learning outcomes compared to the students in the traditional lecture approach’.

Those improvements are due to different reasons, like the variation of studying and learning times and the student-teacher dynamics, that we will deepen later.

In this precise historical period, the issue of blended learning is one of the hottest as regards the school landscape: the emergency of COVID19 that has affected the entire planet has prompted the European community to draw up guidelines for the 2020/2021 academic year to allow a resumption of school activities that respects the health indications issued by the WHO and by the national control bodies.

In these guidelines, however, it is made explicit that the changes that must be made not only for the purpose of allowing a safe resumption of lessons, but also for revolutionizing standard learning as we know it, trying to develop a learning based on skills that is much more student-centred and less disciplined-centred.

During this chapter will be presented the main points that characterize blended learning, what are the research results regarding its application in schools and what are the difficulties and problems encountered in the implementation of such a system.

4.2 Mixed Learning Taxonomy

The term 'Blended' has been used in the last twenty years, together with other terms like 'Hybrid', 'Flipped' or 'Inverted', to denote those courses that merged face-to-face and online instructional methods. In order to understand what the differences between these terms are and what are the modalities in which a blended course can be developed, the MIX (Mixed Instructional eXperience) taxonomy from Margulieux, McCracken and Catrambone (2016) is presented and described. Moreover, the activities presented in the fifth chapter will have this taxonomy as a subdivision to develop blended modules.

According to the MIX taxonomy, four dimensions are identified in order to define and categorize mixed instruction courses. These are:

1. Instructional location (home vs classroom or else);
2. Delivery medium (person vs technology);
3. Instruction type (receiving content vs applying content);
4. Synchronicity (group pace vs individual pace) (Margulieux et al., 2016, p. 105).

Between these four dimensions 'delivery medium' and 'instruction type' are used for the taxonomy. Putting the extremes of each dimension on an axis, four different instructional experiences are found, as shown in Figure 1.

On the left side corners, we have Instructional-transmitted and Technology-transmitted: in the first one the instructional support is delivered primarily via instructor, while in the second via technology. In both of these two modalities the student receives content.

On the right-side corners, we have Instructor-mediated and Technology-mediated: with respect to the left side this time students apply content.

Between the corners there are the so-called combinations, that are *face-to-face combo*, *online combo*, lecture *hybrid* and *practice hybrid*. In the *face-to-face combination* there is the pairing of instructor-mediated and instructor-transmitted. In the *online combination* there is the pairing of technology-mediation and technology-transmitted. *Lecture hybrid* describes courses in which students receive content both via an instructor and via technology, while in *practice hybrid* students apply content under the guidance of both instructor and technology.

In the middle of the axes there are the mixed instruction course, organized in three groups:

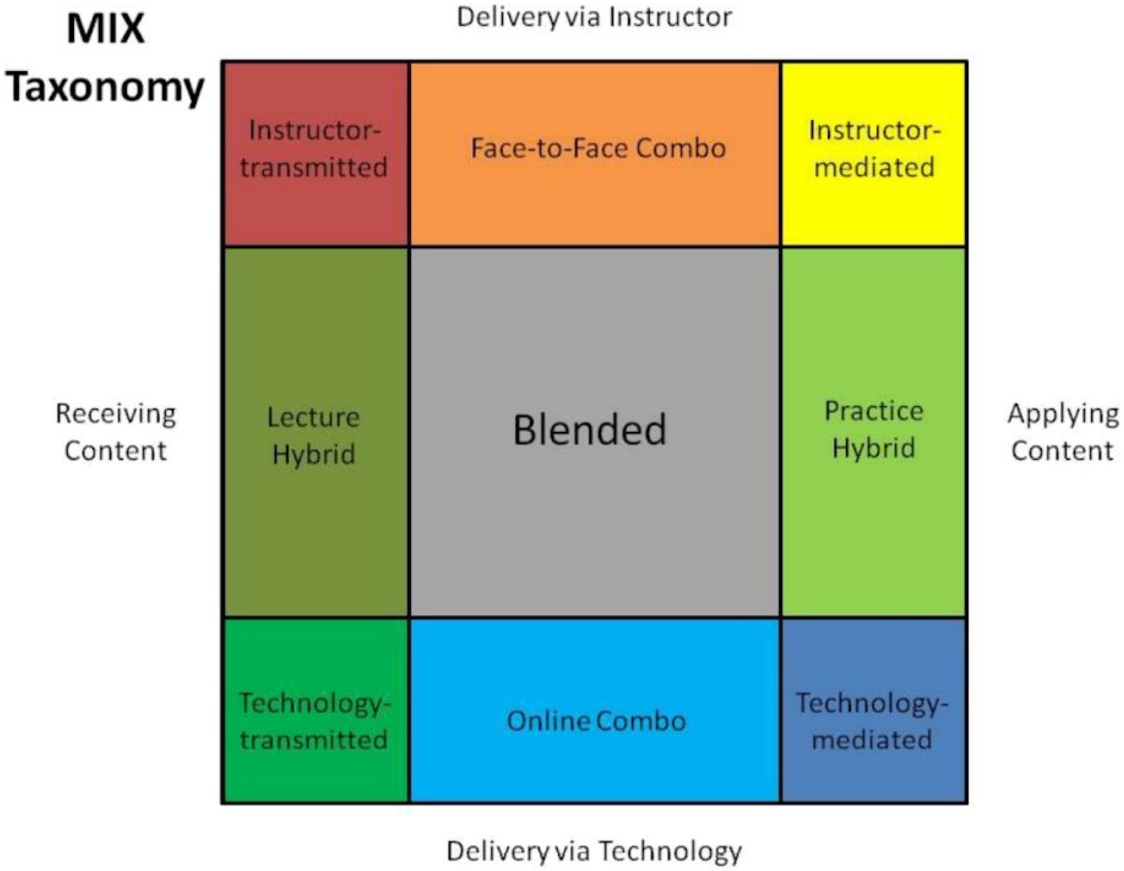
1. Combination (instructional support both for receiving and applying content);
2. Hybrid (delivery of instruction via instructor and via technology);
3. Blended (delivery of instruction via instructor and via technology and provides support during applying and receiving content) (p. 111).

In the Blended group they distinguish three common types of Blended, that are:

- Flipped blend (delivers exposition of content online and delivers feedback on application of content face-to-face);
- Supplemental blend (delivers exposition of content face-to-face and delivers feedback on application of content online);
- Replacement blend (delivers exposition of content and feedback on application of content both face-to-face and online) (p. 111).

Throughout this taxonomy it is possible to classify and distinguish different types of Blended and Mixed learning, in order to understand what the main characteristics of these kinds of approaches are.

Figure 1 - Mixed Instructional eXperience Taxonomy from Margulieux et al. (2016), p. 110



4.3 Blended Learning Main Features

After describing the taxonomy of reference, it is interesting to present what are the main characteristics that distinguish an effective blended approach. For Carman (2005) there are five key ingredients for an effective blended learning:

1. Live events;
2. Multimedia contents;
3. Collaborations;
4. Assessments;
5. Reference materials.

4.3.1 Live events

The first element necessary for the construction of an effective blended approach is the presence of live events, i.e., events in which all students participate at the same time. These events are led by a teacher and require the sharing of a common space where all students can interact with each other, such as a classroom or a virtual classroom.

The path we are trying to follow today, that of Blended Synchronous Learning, involves the simultaneous use of the two places through videoconferencing or web-conference systems, in such a way as to allow all those who cannot be present in the class to succeed to follow the lesson like their peers. This is a big step in the direction of connecting students and those who cannot physically participate in classes, for example, sick people who cannot move from home, or people who are very distant from school or university. With the use of a hybrid communication system, all students can then follow the lessons synchronously and interact with each other and/or with the teacher to ask for explanations or clarifications.

Being able to maintain a synchronous learning element even without necessarily requiring physical presence is essential especially for younger students, who have a greater need to be among their peers: even in the case of physical impediments, developing an effective Blended Synchronous Learning helps these students to take advantage of the functionalities of digital platforms without losing the positive aspects of face-to-face learning, such as the sense of inclusion with peers and the possibility of having instant feedback from the teacher.

4.3.2 Multimedia contents

The second element necessary for the implementation of an effective blended system concerns the development of platforms and services that can be used by students independently and individually. This, more than the others, is the key point without which effective blended learning is impossible.

Developing an easy-to-use and clear digital environment helps students focus on what are the key concepts they need to learn. In this digital environment the student, through multimedia materials such as video-lessons, applets, or podcasts, has more freedom and more time to work on concepts, without necessarily having to maintain the same rhythms as the class; at the same time, by decreasing the number of hours of frontal lessons and working personally with interactive programs, he becomes active with respect to the problems that needs to be solved.

Using these materials, available 24 hours a day, the student has the opportunity to focus on the topics that are most difficult for him, while he can gloss over those in which he feels strongest. In a standard class, where only frontal learning is in force, this is impossible: the teacher must be careful not to deal with too complex topics in order not to lose those students who need more time to understand the main steps; at the same time, however, the lessons must be equally captivating and interesting, in order not to lose the attention of students who show that they have already mastered the concepts.

Through the development of materials that can be used asynchronously and independently, each student can focus on what they prefer; in addition, the teacher can provide different materials according to needs, so as to get the most out of each individual student. Providing functional materials of different types exposes the student to a variety of methods that encourage exploration and independence, according to the constructivist approach.

4.3.3 Collaboration

The third key ingredient for building an effective blended learning environment is collaboration. Both the synchronous sharing of live events and the asynchronous use of online materials become more effective when there is a possibility of meaningful collaboration. While in a classical frontal learning the students are generally passive with respect to the notions provided by the teacher, in a blended learning the students are encouraged to collaborate with each other using multimedia platforms that allow direct communication even outside class hours.

Thanks to the presence of discussion forums integrated with the multimedia platforms used (Learning Management Systems (LMS), which will be discussed later) and videoconferencing systems (for example Microsoft Teams or Zoom), students can interact both with each other in peer-to-peer mode that with the teachers themselves in peer-to-mentor mode: in the first case the peer-to-peer discussions help the students themselves to understand or try to understand complex aspects of the topics studied together, while in the second the teacher can provide targeted aids to the needs of each individual as already discussed above.

4.3.4 Assessment

Assessment tests are the fourth key ingredient of an effective blended approach.

Digital learning environments are equipped with programs and systems to administer training tests before and after having covered a certain topic in class. These formative tests are composed of a large number of different administrable questions: generally, it is possible to develop databases of questions for each topic with varying difficulty. The results obtained by the students in the previous questions will outline the level of the subsequent tests. In addition, some multimedia platforms allow the development of algorithmic exercises, which can be repeated several times since the data is generated randomly each time. The rapidity of acquisition and control of the data collected through the learning analytics systems allows the teacher to carry out a step-by-step analysis of the feedback and pre- and post-lesson assessments: in this way it is possible to have a clear overview of the progress of students, since if the formative tests related to a certain topic bring negative results, the teacher can modify the teaching in progress and act where it is most needed.

4.3.5 Reference materials

The last fundamental point for the realization of effective blended learning concerns the materials provided to the student by the teacher.

Generally, students who follow a traditional type of teaching hardly ever take advantage of the multimedia resources that textbooks offer as they do not have a specific place to find them. By developing an effective Digital Learning Environment, all information is collected in one place, easily accessible, in which the teacher can upload all the materials that the student can use to achieve a clear understanding of the topics.

These materials, which can be slides used in class, in-depth documents, informative videos on the net or links to external online materials, can help the student build a complete and multidimensional image of a certain concept.

To obtain these results it is important that the materials provided by the teacher to the students are effective and functional: they can be prepared by the teachers themselves or they can be taken online. In the latter case, you can find many online sites that offer a large amount of materials that can be used for a fee or for free, such as Khan Academy.

4.4 Multimedia platforms

In order to use and develop engaging multimedia contents and therefore for the implementation of a blended approach is fundamental to discuss multimedia platforms. The main protagonists of a blended course organization are the LMS, platforms on the internet that play the role of a virtual classroom. An LMS can be used alone for the management of a class, but it can also be accompanied by a set of other specific programs used to evaluate or carry out tasks and exercises. In this way, a digital learning environment is formed, also called Digital Learning Environment, where the student can carry out all his tasks without having to change the platform. Examples of this type of program can be Maple or GeoGebra.

4.4.1 Learning Management System

A Learning Management System is an online multimedia platform through which it is possible to manage the course and provide students with the materials necessary for their study. In addition to these main functions, LMSs can offer administrative information related to the course, discussion forums, training tests with immediate feedback and analysis of statistics (Psycharis et al., 2012). To access an LMS, it is generally necessary to register with credentials provided by the school or university attended: in this way the teacher can check the accesses made by students and check the results that individual students obtain in the training tests carried out on the platform.

Some of the best known and most used LMSs today are for example Moodle, Blackboard and WebCT. In particular, among these, Moodle has proved to be very useful in teaching physics and mathematics, as shown by numerous research results.

The use of Moodle in the context of a university course in Physics, Mathematics and Electronic Engineering as a platform in which to propose multiple choice training tests with a database of over 1000 questions has highlighted the existence of a correlation between the marks obtained in the Moodle tests and the final grade of the course, increasing the chances of successfully taking the final exam by a factor of 3 (López et al., 2016). According to Shurygin and Sabirova (2017), the introduction of e-learning courses in physics increases the quality of the learning process, as the use of this type of teaching allows the teacher to effectively organize the learning process and helps the student in find their way among the various resources: the platform records the access data and the time spent in solving the exercises in personal cards for each student, without further effort on the part of the teacher.

4.4.2 Advanced Computing Environment

An Advanced Computing Environment (ACE) is a software that can be used online and offline that allows to perform numerical and symbolic calculations, create graphical representations in 2 or more dimensions, program mathematical simulations in simple language and connect everything using verbal communication, all in a single worksheet. The operation of ACEs is based on a very accessible programming language, like that of CAS (Computer Algebra System).

Through an ACE, it is possible to develop problem solving skills as this offers various types of representation and allows the resolution of problems that are difficult to solve by hand, thus allowing the student to concentrate on solving the problem and not on calculation difficulties. In addition, there is a wide range of problems that can be solved through such a system, which helps the student to analyse a certain concept from multiple points of view.

A very important part of the ACEs is the design and programming of interactive components: these allow the visualization of the variation of the results obtained in a simulation or in a certain exercise based on the data entered as input. Doing so allows to generalize the process of solving a problem allowing the student to focus on the most important aspects (Di Luca & Marchisio, 2018).

In addition, through the ACEs it is possible to create interactive test sheets that complement the classic textual demonstrations with visuals, complementing everything with questions aimed at verifying the student's learning. Through ACE such as Maple it is possible to carry out demonstrations of physical phenomena that combine mathematical language and graphic language (Barana et al., 2019).

4.4.3 Virtual Lab

In blended learning, the laboratory aspect is also addressed, through the use of simulations and interactive graphic representations. In addition, through these means it is possible to have visual representations of experiments that generally would not be possible, due to lack of means or conditions (Li & Tang, 2017). Developing a virtual laboratory with combinations of movements and illustrations allows you to recreate and demonstrate physical processes, which can stimulate students to think, imagine and innovate. (Li & Tang, 2017).

Combining these virtual laboratories with simulations or applets allows the student to have experiences that would not be feasible without the use of a digital environment. The use of visualizations, simulations and multiple representations helps the student to better understand key physical processes, qualitative experiments and to develop critical reasoning skills.

In this regard, Li and Tang (2017) offer an example of an application of what they call “Active Physics Online” regarding special relativity. Many research results in Physics Education show that special relativity, as well as other topics far from everyday reality (e.g., quantum mechanics or astrophysics), are difficult to understand because due to their intrinsic nature they do not allow a clear representation of phenomena in progress (for example trains traveling at speeds close to c). Through the use of virtual simulations, interactive exercises, and virtual reality it is possible to create systems that help in the visualization of the basic concepts of the theory.

Besson (2015) cites some examples of apps or programs that simulate relativistic effects to allow a concrete visualization of the phenomena in question. In particular, the Real Time Relativity software, developed by researchers in Australia for first-year university students, allows you to view on a screen "the simulation of a three-dimensional world populated by buildings, clocks, streets [...] seen by a stationary observer or in motion at high speeds, which the user can freely modify. [...] By observing the indications of the watches, it is possible to discuss issues related to simultaneity and the chronological order of events. " (p. 260).

These types of simulations are very useful for all those students who prefer a study based on images, videos, and “evidence” rather than one based on theory, definitions, and calculations.

4.5 Methodologies of Blended Learning

There are many ways to implement Blended Learning. Alammary and colleagues (2014) distinguished three different ways in which a blended approach can be introduced in curricula, that are low-impact blend, medium-impact blend, and high-impact blend. In the first modality the instructor adds extra online activities with respect to the standard face-to-face course; in the second one the instructor replaces some of the standard activities online or multimedia activities; in the third on the whole course is designed from scratch with the aim of building an effective blended course in all its aspects. The relevance of each of these three approaches depends on the context, the skills of the teacher, and the goals to be achieved. In the following sections some examples of blended approaches are described, underlying the methodological aspects but also the pedagogical ones.

4.5.1 Adding online activities to traditional courses

The first way in which a blended approach can be built is by adding online activities to a traditional course, i.e., a low-impact blended approach. These activities can have different pedagogical purposes: they can be designed to make students communicate more with each other, or they can be designed to make students focus on a particular topic. An example of this type of approach is the one described by McCarthy (2010).

This kind of low-impact approach has some benefits but also challenges, as described by Alammary and colleagues (2014). The benefits of this approach lie in the fact that by not requiring a very large commitment from teachers even the most reluctant can approach the world of blended learning. In addition, this methodology allows them to make changes quickly without exposing themselves to too much risk of failure.

In terms of challenges, teachers need to have some technological knowledge to be able to effectively apply this approach. In addition, there is the risk of creating a course in which extra activities are added to an already complete module, going to overload students and instructors without any real sense.

4.5.2 Teaching 2.0

‘Teaching 2.0’ is an example of medium-impact blend. With this term usually is described an instructional approach based on intensive use of Information and Communication Technologies (ICT). This approach is also linked to other innovative teaching methodologies, such as cooperative learning, project-based teaching, metacognitive approaches, and laboratory teaching.

In order to develop an environment based on Teaching 2.0 there are some necessary elements:

- use of digital devices by students (like laptop or tablet);
- use of technological devices to share what has been done by students (like projectors or Multimedia Backboard);
- a wireless connection whit which students can cooperate online;
- virtual spaces where students can share contents and results of their works;
- structuring of flexible classroom space in relation to different work situations (group, individual, etc.);
- use of both textbooks and online resources (eBook, ePubs, etc.).

In this approach the teacher must design and coordinate the setup of the learning environment, but is not obliged to revolutionize the traditional course. The differences with traditional didactics lie in the ability to sit alongside the student to accompany them in the use of technological and cultural resources. In this way, the student experiences a relationship with cultural content that is more similar to the one he or she has in everyday life (Castoldi, 2014).

4.5.3 Flipped Classroom

An example of high-impact blend is the Flipped Classroom. This model was born in the early 2000s thanks to the contribution of numerous teachers around the world and the birth of online sharing services for multimedia material such as audio and video. In particular, it is emblematic to present the case of two Colorado chemistry professors, Bergmann and Sams (2012).

In 2007, to allow all the students of their courses not to miss lessons, recorded the lessons that would then be held in class and made them available on the internet. After receiving numerous positive feedback about the videos, Bergmann and Sans decided to pre-record all the lessons of the chemistry course and assign as homework to watch one of these lessons before the actual class meeting. In this way the students had the possibility to stop the video and to watch it several times if a certain topic was not clear, or to continue those parts that appeared already understood. In addition, students were asked to take notes on the aspects they found most difficult. Once in the classroom, the students took the time together to talk with the teacher on these aspects and to carry out discussions and exercises with the whole class. The results of applying this method have shown that students are more satisfied than in classic lessons as they are not forced to passively undergo a frontal lesson but have the possibility and duty to interact with their classmates and the teacher himself, transforming top-down to peer-to-peer teaching.

According to Cecchinato (2012), the Flipped Classroom method has "multiple pedagogical implications: from the individualization and personalization of learning to active and peer learning, making it possible to transform a fundamentally instructional teaching into a constructivist and social one" (p. 11).

4.6 Blended Learning Issues

While there are many positive aspects of a blended approach, there are challenges that need to be addressed in order to get the most out of this type of teaching and learning. A review of all these challenges and issues was made by Rasheed, Kamsin and Abdullah (2020), with the aim of identifying the challenges that in particular concern the online component of blended learning from the perspective of students, teachers, and educational institutions.

According to Rasheed, the essential elements for blended teaching to be effective are 3:

- that students are able to self-regulate and have technological skills that allow them to organize and carry out their studies independently from the teacher, at their own pace, and to manage online technologies that allow study separate from lectures;
- that teachers are technologically competent, in order to use and manage the technological tools for teaching, and also to create and upload online the materials needed by students (for example, recorded lessons or quality videos);
- that schools and institutions in general provide the necessary preparation and adequate technological support to both students and teachers to ensure effective use of the technological tools available, and also to organize and effectively use the online component of the communication system used.

In the study mentioned above, the main problems of each of these three macro-sectors are then listed: focusing on the first of these three, namely that of students, the main problems concern self-regulation in the time of individual study (i.e. asynchronous study), the lack of technological skills in the management and use of the digital systems used, the isolation that arises from individual study, the high complexity that some technologies may require and the possible lack of suitable or adequate technological devices.

The latter problem is due to the fact that some students may not have access to the online resources provided by the teacher or the school for issues related to social or economic status or simply to the availability of PCs or electronic devices if other family members have the same needs, like for example a family with two children and only one personal computer.

To overcome these problems, schools and universities generally provide school computers or sim cards with internet connections to allow all students to have the same quality of training. In this sense, the goal to be achieved is that of a global digital equity, defined by the National Digital Inclusion Alliance in 2019 as:

"A condition in which all individuals and communities have the information technology capacity needed for full participation in our society, democracy and economy. Digital Equity is necessary for civic and cultural participation, employment, lifelong learning, and access to essential services. " (https://www.digitalinclusion.org/definitions/).

As far as teachers are concerned, one of the difficulties that can arise in the transition to blended teaching is that of classroom management: in order to carry out this type of teaching in an effective and functional way, the teacher must be able to manage both students in presence than those connected remotely, without however negatively affecting the quality of teaching. For example, students in the classroom could feel forced by the use of digital systems with which to interact, while to make up for the absence of a part of the class the teacher could fall back on teaching based on non-specific slides or videos, which could not be effective if not studied and prepared in detail.

In addition, it is important to keep in mind that to carry out an online teaching it is not enough to perform the classic lessons that would be done in class on the computer, but it is necessary to reformulate both the times and the strategies to ensure that both those who follow from home and those who follow in class have the same opportunities.

From this point of view, over the years, different teaching methods have been developed that provide for a side-by-side classic frontal learning with learning based on the use of electronic and technological devices. Among these, the flipped classroom technique is very close to the idea of blended learning that has been presented so far.

4.7 Conclusions

In conclusion, the numerous articles analysed tell us that blended learning helps students to build a deeper knowledge of concepts, with a consequent improvement in the results obtained. As shown by Margulieux and colleagues (2016), “courses that used mixed instruction to reduce time spent in class by delivering part of instruction online maintained equivalent learning outcomes while reducing time spent in class”, while “courses that used mixed instruction to start providing feedback while students applied content improved learning outcomes, while commonly maintaining the time that students spent on a course” (p. 115).

However, in order to implement an effective and functional system, in addition to the high organizational skills required of the teacher, the collaboration of the entire school system is necessary, starting from the administration passing through the teachers and arriving at the students, since “the short history of online learning is filled with carcasses of projects where misunderstandings between administration and faculty have never been resolved” (Moskal et al., 2013).

In the next chapter a module composed of 4 activities based on the analysis carried out in the third chapter is presented. These activities are designed to be implemented in a medium/high impact blending, using both the Flipped Classroom method (Activity #1) and the Teaching 2.0 method (Activity #3).

Chapter 5 - Activities

5.1 Introduction

In this chapter will be presented a series of activities based on the analysis made in the third chapter. These activities are thought to be implemented as a part of an IDENTITIES module that will be tested next year.

The place in which these activities will be realized is a pre-service teacher formation course about physics, mathematics, informatics, and science education.

The activity is designed to be carried out in a blended modality and will be divided in several parts:

- Introduction to the topics and to the interdisciplinary framework (video lessons, synchronous or asynchronous):
- Analysis of the excerpts from the original papers with itinerant questions (group work in synchronous modality) and discussion on the new images of Special Theory of Relativity (STR) that come out from the historical analysis and on the differences between this new approach and those proposed by normal textbooks (synchronous activity);
- Different ways to derive Lorentz Transformations (LT) following the reasoning of Poincaré (groups of transformations), Einstein (logical reasoning from postulates) and Minkowski (geometric reasoning) (synchronous group work);
- Discussion on the implications that this new approach have on special relativity (group discussion) and individual essays on the benefits that a similar approach can bring to the understanding of the theory, of the role played by LT and on the image of interdisciplinarity (individual essay).

5.2 Activity #1 – The 1900s and the need for Interdisciplinarity

The first part of this activity is made with the goal of introducing the topics that will be studied and the interdisciplinary framework used to analyse them. This activity is designed to be realized with a Flipped approach, i.e., a high-impact blend.

This part consists of 3 or 4 video lessons of about 10/15 minutes in length, each of them with a different topic.

1. The first one will focus on the description of the historical period of the late 1900s in the physics field. The video can be made on purpose for this activity or other video already existing can be used. The goal of this video lesson is to explain which were the reasons that led to the revolution of the Special Theory of Relativity, and why the Special Theory of Relativity was not realized before despite the fact that the mathematical bases were already present. An overview of the main discoveries and revolutions made in this century, like the Maxwell theory of electrodynamics, Boltzmann theory of Thermodynamics or Röntgen X-Rays. A point to be stressed is that even if the mathematical structure of the LT had already been found in 1887 by Voigt, what was missing were some experimental results that proved the inconsistency of the ether dragging theory, like the Michelson and Morley one.

2. The second video will focus, in a mirror manner to the first, on the revolutions that occurred in the nineteenth century in the field of mathematics.
   In this video there will be a presentation of the major changes that shaped the mathematics of this century and in this way put the basis for the development of several research fields still open today. An example for these changes is represented by the birth of Non-Euclidean Geometries following the works of Gauss, Lobachevsky, and Bolyai, or by the 'Erlangen Program' from Felix Klein and the introduction of the concept of group made by him and lie.

   These two videos can be merged in a single video lesson due to the interplay between the discoveries made in the century. Merging the video can be a useful way to show how deep was the connection between the two disciplines and how this connection played a central role for the development of science as we know it today.

3. The third video will focus on the presentation of the original papers from Lorentz, Poincaré, Einstein and Minkowski. This presentation is made with the aim of showing the contents of the papers and the differences between them, from many points of views: contents, linguistic, methodological, and epistemological. There is also the possibility to make four separate videos, one for each authors, so that they can be used again in the activity 2.

4. The fourth video lesson will focus on the presentation of the interdisciplinary framework of "Boundary Crossing and Boundary Object" from Akkerman and Bakker. Starting from an overview of their paper, the stress will be made on the concept of Boundary, as intended by them but also by other different perspectives (scientific, social, anthropological).
   Then there will be a discussion about the four different learning mechanisms presented in the paper, that are identification, coordination, reflection, and transformation. After this there will be a part of the video in which there is the presentation of how these learning mechanisms have been considered and applied in the IDENTITIES project.

These videos are made with the aim of presenting the elements necessary for the full understanding of the work that will be done in the other parts of the activity. Moreover, in this way this activity can be developed in many different situations (for example a physics education course or a mathematics education one) even if the responsible for the course is not an expert in that field.

The video lessons can be seen together in class in a synchronous way or individually at home in an asynchronous way. In this way every student has the possibility of following the lessons at his own path, and if something is not clear he can go back with the lesson or search for some more information in the material that the teacher will give him as course material. This material will include the original papers, the relative part of this thesis and other documents or videos considered important for the full understanding of the topics considered.

This first activity will occupy between one and two hours, being the various videos of 15 minutes each.

5.3 Activity #2 – Analysis of the original papers

In the second activity there will be an analysis made by the students using some itinerant questions based on the analysis already made in the third chapter with the goal of guiding the students to highlight the learning mechanisms present in the various papers using the interdisciplinary framework explained previously.

This activity can be carried out in small groups of 4 or 5 students which can work online or in a face-to-face modality, i.e., a low-impact blend. Every group will work on a couple of excerpts taken from different papers. In this way this second activity will have 3 different parts:

1. analysis of the first excerpt with 4 or 5 itinerant questions;
2. analysis of the second excerpt with the same questions used for the previous excerpt;
3. comparison of the results obtained from the questions of the first and second part and discussion on the differences obtained. This discussion can be carried out orally or individually with a set of questions that need to be answered in the form of a small essay.

The combination of the excerpt should be made so that every group will analyse the excerpts from two different authors but at the same time every author has to be analysed at least by two different groups. If possible, the analysis should be made in such a way that there is a comparison between every author. The final combination will be, if for example there are 30 students in the class and every group is composed by 4/5 students:

- Group 1 - Lorentz and Poincaré excerpts;
- Group 2 - Lorentz and Einstein excerpts;
- Group 3 - Lorentz and Minkowski excerpts;
- Group 4 - Poincaré and Einstein excerpts;
- Group 5 - Poincaré and Minkowski excerpts;
- Group 6 - Einstein and Minkowski excerpts.

Every section of this first part of the second activity should last 30/45 minutes. Totally this part will occupy 2/2.5 hours.

Once that the comparison within the groups is made there will be a presentation of the results obtained by the single groups. This discussion will allow the different groups to share their different results so that there will be a fruitful dialogue between different opinions.

At this point a comparison with the results obtained in the third chapter will be made, in order to see how the interdisciplinary lenses were applied by the different groups with respect to the work made in the thesis.

The presentation of the results and the discussion should last at least one hour.

5.4 Activity #3 – Obtaining the Lorentz Transformations

The goal of the third activity is to obtain LT in two different ways, following the works made by Einstein and Minkowski. There is the possibility of obtaining LT also starting directly from Maxwell Equations in order to find the Transformation Group of the equations (that is the Lorentz Group) but this method can be more difficult with respect to the others depending on students' prior knowledge and field of study.

This activity is made to show the different roles that LT have in the different papers and how this different role is able to highlight a different aspect of them. In this way the nature of LT as a boundary object will emerge.

- The first way to obtain LT is based on Einstein work made in his 1905 paper, where he obtains LT as a logical result of the Relativity Postulate and the constancy of c.
- The second way to obtain LT is based on Minkowski's geometrical approach. In his work he obtains the Transformations as a geometrical result that comes from changing the reference system using the hyperbola as a gauge for transformations.

To accomplish this goal a guide will be made with the aim of showing which are the passages that have to be made in order to obtain the transformations. These passages will be realized using digital resources, like ready to use exercises developed on a program like GeoGebra or Maple that step after step will show to the students which are the results obtained in every part of the process.

This activity will be realized in groups of 3/4 students and should last about 1 hour and due to its design is categorized as a medium-impact blend.

5.5 Activity #4 – STR' Interdisciplinarity Implications

In this final activity a discussion about the implications that this new approach to the Special Theory of Relativity have for the vision that we have about the interdisciplinarity between mathematics and physics. Moreover, in this final part every student will have the possibility of showing what this new approach brought to their understanding of the theory.

This activity is divided into two parts. The first part is a discussion made in the class in which the teacher assumes the role of a defiladed guide, while the students will be the architects of the discussion.

In this discussion a starting point will be set by the teacher. This starting point is a question like: "What do you think this interdisciplinary approach to STR can bring to the conversation?".

After the discussion, each student will be asked to report their ideas with a questionnaire. In this questionnaire, to which they can answer individually in the week following the activities, each student will answer some starting questions developed along the lines of those addressed in the common discussion. An example of these questions can be:

- In your opinion, this interdisciplinary approach to STR adds some new value to the standard conversation about the theory? Which are the differences that you noticed between this approach and the way you learned STR?
- In your opinion, have the activity made on LT helped you to understand them in a deeper way?
- Which image of interdisciplinarity comes to your mind after this analysis?

In the questionnaire there will be for the students also the possibility of writing their own thoughts on the module. This free part does not have a limit in length and its analysis will be helpful to understand if the activities were valid for a deeper understanding of the theory.

Chapter 6 - Conclusions

The objective of this thesis was to devise activities to be carried out in blended mode that would highlight the intrinsic interdisciplinary nature of STR and the multiple roles played by Lorentz Transformations (LT) i

The design of the activities has been framed within a wide literature review that has been carried out to highlight which are the most fertile areas of research on Special Theory of Relativity (STR) in PER (Physics Education Research) and MER (Mathematics Education Research). The results of this review showed that there are 7 main categories of research regarding STR in PER, that are Student Difficulties, Digital Tools Development, History and Epistemology, Curricula Development, Conceptual Change, In/Pre-Service Teachers Formation, and Interdisciplinarity. The most addressed strands are Student Difficulties and Digital Tools Development, while the least addressed is Interdisciplinarity. As far as the MER field, the biggest strand is the ones that focuses on the geometrical aspects of STR from an interdisciplinary point of view, where the concept of group transformations is stressed as fundamental to foster deep and meaningful understanding of the basics aspects of the theory.

From the review, hence, emerged that interdisciplinarity is not a topic widely explored and analysed. Also, LT are studied usually as an instrument that helps students making demonstrations in order to obtain Relativistic Effects, but what is missing from the review is an analysis of the different roles that LT have played in the genesis of the theory and can play today to unpack the rich relation between mathematics and physics.

To comply with these shortcomings, in chapter three the results of an interdisciplinary analysis of the original STR texts is reported. The analysis has been carried out by applying interdisciplinary lenses from the research frame of "Boundary Crossing and Boundary Object" introduced by Akkerman and Bakker (2011).

The results of this analysis showed that in the four historical articles are four different ways of seeing the relation between mathematics and physics: in the Lorentz paper it can be recognised the learning mechanism of Coordination between mathematics and physics, where the Lorentz Transformations act as a boundary object without any deep epistemic change in either of the two disciplines; in Poincaré work the learning mechanism acting between the two disciplines appears as a process of Transformation, in that the final result of his work is a merging of the two disciplines into one unique approach; in the third work by Einstein, the predominant learning mechanism that can be acknowledged appears Reflection, since he applies the logical reasoning of mathematics to the physical world and, from the “eyes” of that perspective, the physics way to look at reality changed; in Minkowski’s paper the learning mechanism that can be recognised is, again, Transformation, in that the merging of geometry and physics gave birth to the concept of space-time and this led to the foundation of what would have then been general relativity.

Talking about LT, their nature of boundary object emerges strongly from this analysis. LT can be seen under a different light in each of the four works considered, and in every work, they bring out a different vision of the theory. They are at the same time a mathematical aid to prove experimental results, a set of equations demonstrated by the Least Action Principle, a group of transformations, the logical results of the Relativity Postulate and the constancy of c, and the geometrical result that comes from changing the reference system using the hyperbola as a gauge for transformations.

This multiple vision made evident by the analysis was then used to design a series of activities, four to be exact, that will complete a module that is part of the Erasmus+ IDENTITIES project that will be tested next year. These activities have been designed to be carried out in blended mode according to the indications described in chapter four.

Each activity is designed to cover a different aspect of the content covered in this thesis and to be carried out following different blending methods.

The first activity is designed to be carried out using a Flipped approach, and aims to introduce students to the scientific landscape of the late XIX century and early XX century, the contents of the original STR papers, and the interdisciplinary frame of the Boundaries.

The second activity is designed to be realized using a low-impact or medium-impact blending approach, and aims to have students, divided into groups, analyse excerpts taken from the original STR articles to let them experience first-hand the differences in approaches between the different authors.

The third activity use a medium-impact/high-impact blend approach. In this activity students, using Advanced Computing Environments (ACE), trace the steps that led Einstein and Minkowski to obtain LT. These two approaches were chosen to show the vast differences between the two authors and the multiple nature that LTs can hold in theory.

The fourth and last activity has the objective of getting students to talk to each other, to make them reflect on the path they have taken and what added value, if any, this new approach has given them in their understanding of the theory. This activity, due to its goal, is designed to be realized using a low impact blending approach, as it is preferable that the comparison be conducted in large groups or preferably all together. In order to obtain feedback on the module, online questionnaires will be administered and then evaluated and analysed once the module is over.

Appendix

Lorentz Excerpts (Lorentz, 1904)

Section 2 (pp. 810-11)

The experiments of which I have spoken are not the only reason for which a new examination of the problems connected with the motion of the Earth is desiderable. Poincaré has objected to the existing theory of electric and optical phenomena in moving bodies that, in order to explain Michelson's negative result, the introduction of a new hypothesis has been required, and that the same necessity may occur each time new facts will be brought to light. Surely this course of inventing special hypotheses for each new experimental result is somewhat artificial. It would be more satisfactory, if it were possible to show, by means of certain fundamental assumptions, and without neglecting terms of one order of magnitude or another, that many electromagnetic actions are entirely independent of the motion of the system. Some years ago, I have already sought to frame a theory of this kind. I believe now to be able to treat the subject with a better result. The only restriction as regards the velocity will be that it be smaller than that of light.

Section 4 (pp. 812-813)

We shall further transform these formulae by a change of variables. Putting $\frac{c^2}{c^2-w^2} = k^2$ *and understanding by l another numerical quantity, to be determined further on, I take as new independent variables*

$$x' = klx, y' = ly, z' = lz, t'^{\left(\frac{l}{k}\right)}t - kl\ \left(\frac{w}{c^2}\right)x.$$

As to the coefficient l, it is to be considered a function of w, whose value is 1 for w=0, and which, for small values of w, differs from unity no more than by an amount of the second order.

The variable t' may be called the "local time"; indeed, for $k = 1, l = 1$ *it becomes identical with what I have formerly understood by this name.*

Section 8 (pp. 818-819)

Thus far we have only used the fundamental equations without any new assumptions. I shall now suppose "that the electrons, which I take to be spheres of radius R in the state of rest, have their dimensions changed by the effect of a translation, the dimensions in the direction of motion becoming kl times and those in perpendicular directions l times smaller". In this deformation, which may be represented by $\left(\frac{1}{kl}, \frac{1}{l}, \frac{1}{l}\right)$, *each element of volume is understood to preserve its charge.*

Our assumptions amounts to saying that in an electrostatic system Σ, moving with a velocity w, all electrons are flattened ellipsoids with their smaller axes in the direction of motion. If now [...] we subject the system to the deformation (kl, l, l) *we shall have again spherical electrons of radius R. Hence, if we alter the relative position of the centres of the electrons in Σ by applying the deformation* (kl, l, l), *and if, in the points thus obtained, we place the centres of electrons that remain at rest, we shall get a system, identical to the imaginary system Σ'. [...] In the second place I shall suppose "that the forces between uncharged particles, as well as those between such particles and electrons, are influenced by a translation in quite the same way as the electric forces in an electrostatic system". In other terms, whatever be the nature of the particles composing a ponderable body, so* *long as they do not move relatively to each other, we shall have between the forces acting in a system (Σ') without, and the same system (Σ) with a translation, the relation* $F(\Sigma) = \left(l^2, \frac{l^2}{k^2}, \frac{l^2}{k^2}\right) F(\Sigma')$, *if, as regards the relative position of the particles, Σ' is got from Σ by the deformation* (kl, l, l),, *or Σ from Σ' by the deformation* $\left(\frac{1}{kl}, \frac{1}{l}, \frac{1}{l}\right)$.

We see by this that, as soon as the resulting force is 0 for a particle in Σ', the same must be true for the corresponding particle in Σ. Consequently, if, neglecting the effects of molecular motion, we suppose each particle of a solid body to be in equilibrium under the action of the attractions and repulsions exerted by its neighbours, and if we take for granted that there is but one configuration of equilibrium, we may draw the conclusion that the system Σ', if the velocity w is imparted to it, will of itself change into the system Σ. In other terms, the translation will produce the deformation $\left(\frac{1}{kl}, \frac{1}{l}, \frac{1}{l}\right)$.

It will easily be seen that the hypothesis that has formerly been made in connexion with Michelson's experiment, is implied in what has now been said. However, the present hypothesis is more general because the only limitation imposed on the motion is that its velocity be smaller than that of light.

Poincaré excerpts (Poincaré, 1906. Translated by Popp, 2020)

Introduction (pp. 45-46)

On first consideration it seemed that the aberration of light and the optical and electrical phenomena associated with it were going to provide us a means for determining the absolute motion of the Earth or more accurately its motion, not with respect to the other stars, but with respect to the ether. Fresnel had already tried it, but he soon recognized that the motion of the Earth did not change the laws of refraction and reflection. Analogous experiments, like that of the water-filled telescope and all those where only first-order terms in the aberration were considered were to give only negative results; the explanation for this was soon found. But, Michelson, who had imagined an experiment sensitive to the terms depending on the square of the aberration, failed in turn. It seems that this impossibility of showing the absolute motion of the Earth experimentally could be a general law of Nature; we are naturally led to accept this law, that we will call the Relativity Postulate and to allow it without restriction. Should this postulate, until now in agreement with experiment, later be confirmed or rejected by more precise experiments, it is in any case interesting to look at what its consequences might be. An explanation was proposed by Lorentz and Fitz Gerald, who introduced the hypothesis of a contraction experienced by all bodies in the direction of motion of the Earth and proportional to the square of the aberration; this contraction, which we will call the Lorentz contraction, took into account the Michelson experiment and all those which had been done until now. The hypothesis would become insufficient, however, if the relativity postulate were to be accepted in its full generality.

Lorentz sought to supplement it and amend it so as to bring it into full agreement with this postulate. This is what he succeeded in doing in his article entitled Electromagnetic Phenomena in a System Moving with Any Velocity Smaller than that of Light (Proceedings of the Amsterdam Academy, May 27, 1904). The importance of the question led me to take it up again; the results that I obtained are in agreement with those of Lorentz on all important points; I was only led to amend and supplement them in some points of detail. The differences, which are of secondary importance, will be seen later. Lorentz's idea can be summarized as follows: if one can, without any visible phenomenon being modified, give any system a shared translation, it is because the equations of the electromagnetic environment are not altered by certain transformations, which we will call Lorentz transformations; two systems, the one stationary and the other in translation, thus become the exact image of each other. Langevin had sought to modify Lorentz's idea; for both authors, the moving electron takes the form of a flattened ellipsoid, but for Lorentz two of the axes of the ellipsoid remain constant and in contrast for Langevin it is the volume of the ellipsoid which remains constant. Both authors additionally showed that these two hypotheses agree with Kaufmann's experiments and also with Abraham's primitive hypothesis (undeformable spherical electron). The advantage of Langevin's theory is that it does not call on electromagnetic forces and binding forces; but it is incompatible with the relativity postulate. That is what Lorentz had shown; it is what I found in turn by another route by calling on the principles of group theory. That means it is necessary to go back to Lorentz's theory; but to keep it and avoid intolerable contradictions, a special force has to be assumed which explains both the contraction and the two constant axes. I sought to determine this force, and I found that it could be compared to a constant external pressure acting on the deformable and compressible electron and its work is proportional to the variations in the volume of this electron. If the inertia of matter were then exclusively of electromagnetic origin, as is generally accepted since Kaufmann's experiment, and if all the forces are of electromagnetic origin other than this constant pressure that I just spoke of, then the relativity postulate can be established with full rigor. That is what I show by a very simple calculation based on the principle of least action.

Section 1 (pp. 48-53)

Lorentz adopted a specific system of units so as to make the factors of 4π disappear from the formulas. I will do the same and additionally I will choose the units of length and time such that the speed of light is equal to one. Under these conditions, by calling: f, g, h the electric displacement; α, β, γ the magnetic force; F, G, H the vector potential; ψ the scalar potential; ρ the electric charge density; ξ, η, ζ the electron velocity; and u, v, w the current, the fundamental formulas become:

$$u = \frac{df}{dt} + \rho\xi = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \alpha = \frac{dH}{dy} - \frac{dG}{dz}, f = -\frac{dF}{dt} - \frac{d\psi}{dx}, \frac{d\alpha}{dt} = \frac{dg}{dz} - \frac{dh}{dy}, \frac{d\rho}{dx} + \Sigma\frac{d\rho\xi}{dx} = 0,$$

$$\Sigma\frac{df}{dx} = \rho, \frac{d\psi}{dt} + \Sigma\frac{df}{dx} = 0, \square = \Delta - \frac{d^2}{dt^2} = \Sigma\frac{d^2}{dx^2} - \frac{d^2}{dt^2}, \psi = -\rho, F = -\rho\xi.$$

An element of matter of volume $dxdydz$ experiences a mechanical force whose components $Xdxdydz$, $Zdxdydz$, $Ydxdydz$ are determined from the formula:

$$X = \rho f + \rho(\eta\gamma - \zeta\beta).$$

These equations are subject to a remarkable transformation discovered by Lorentz, and which is of interest because it explains why no experiment is able to let us know the absolute motion of the universe. Let us set:

$$x' = kl(x + \varepsilon t), t' = kl(t + \varepsilon x), y' = ly, z' = lz.$$

where l and ε are arbitrary constants, and where

$$k = \frac{1}{\sqrt{1-\varepsilon^2}}.$$

If we then set:

$$\square = \Sigma\frac{d^2}{dx'^2} - \frac{d^2}{dt'^2}$$

it will follow:

$$\square' = \square l^2$$

Now consider a sphere driven in a motion of uniform translation with the electron and let:

$$(x - \xi t)^2 + (y - \eta t)^2 + (z - \zeta t)^2 = r^2$$

be the equation of this mobile sphere whose volume will be 4/3 πr3. The transformation will change it into an ellipsoid whose equation is easy to find. It is in fact easily deduced from equations

$$x = \frac{k}{l}(x' - \varepsilon t'), t = \frac{k}{l}(x' - \varepsilon t'), y = \frac{y'}{l}, z = \frac{z'}{l}.$$

The equation for the ellipsoid then becomes:

$$k^2(x' - \varepsilon t' + \varepsilon\xi x')^2 + (y' - \eta k t' + \eta k\varepsilon x')^2 + (z' - \zeta k t' + \xi k\varepsilon x')^2 = l^2 r^2$$

This ellipsoid moves with a uniform motion; for $t' = 0$, it reduces to

$$k^2 x'^2 (1 + \varepsilon\xi)^2 + (y' + \eta k\varepsilon x')^2 + (z' + \xi k\varepsilon x')^2 = l^2 r^2$$

and its volume is

$$\frac{4}{3}\frac{\pi r^3 l^3}{k(1 + \xi\varepsilon)}$$

[...]

If the inertia of the electrons is exclusively of electromagnetic origin and if additionally they are only subject to forces of electromagnetic origin, then the equilibrium condition requires that inside the electrons it hold: $X = Y = Z = 0$. Hence [...] these relations are equivalent to $X' = Y' = Z' = 0$.. The equilibrium conditions of the electrons are therefore unchanged by the transformation. Unfortunately, such a simple assumption is not allowable. If, in fact, one supposes that ξ = η = ζ = 0, the conditions $X = Y = Z = 0$ would lead to $f = g = h = 0$, and consequently $\Sigma\frac{df}{dx} = 0$, meaning ρ = 0. One would arrive at analogous results in the most general case. One therefore has to accept that in addition to electromagnetic forces, there are either other forces or binding.

One must then look at what conditions these forces or binding must satisfy for the equilibrium of the electrons to be undisturbed by the transformation. This will be taken up in a subsequent section.

Section 4 (pp. 64-67)

We are therefore led to consider a continuous group that we will call the Lorentz group in which will allow as infinitesimal transformations:

1) the transformation T_0 which will be permutable with all the others;

2) the three transformations T_1, T_2, T_3;

and 3) the three rotations $[T_1, T_2]$, $[T_2, T_3]$, $[T_3, T_1]$.

where

$$T_0 = x\frac{d\varphi}{dx} + y\frac{d\varphi}{dy} + z\frac{d\varphi}{dz} + t\frac{d\varphi}{dt};$$

$$T_1 = t\frac{d\varphi}{dx} + x\frac{d\varphi}{dt};$$

$$T_2 = t\frac{d\varphi}{dy} + y\frac{d\varphi}{dt};$$

$$T_3 = t\frac{d\varphi}{dz} + z\frac{d\varphi}{dt};$$

$$[T_1, T_2] = x\frac{d\varphi}{dy} - y\frac{d\varphi}{dx}.$$

An arbitrary transformation of this group could always be broken down into a transformation of the form:

$x' = lx, y' = ly, z' = lz, t' = lt$ *and a linear transformation which does not change the quadratic form:*

$$x^2 + y^2 + z^2 - t^2$$

We can also generate our group in another way. Any transformation of the group could be regarded as a transformation of the form:

$$x' = kl(x + \varepsilon t), y' = ly, z' = lz, t' = kl(t + \varepsilon x) \qquad (1)$$

preceded and followed by a suitable rotation. But for our purposes, we should only consider a part of the transformations from this group; we should assume that l is a function of ε, and it will be a matter of choosing this function such that this part of the group, which I will call P, again forms a group. Turning the system 180 around the y-axis, we should find a transformation which will have to again belong to P. Now this amounts to changing the sign of x, x' , z and z'; in that way it is found that

$$x'=kl(x - \varepsilon t), y'=ly, z'=lz, t'=kl(t - \varepsilon x) \qquad (2)$$

Thus l is not changed when ε is changed to -ε. On the other hand, if P is a group, the inverse substitution of (1), which is written:

$$x' = \frac{k}{l}(x - \varepsilon t), y = \frac{y'}{l}, z = \frac{z'}{l}, \qquad t' = \frac{k}{l}(x - \varepsilon x).$$

should also belong to P; it will therefore have to be identical to (2), meaning that , $l = \frac{1}{l}$.

It will therefore have to be that l = 1.

Einstein excerpts (Einstein, 1905. Translated by Perrett & Jeffery, 1923)

Section 1 (pp. 38-40)

Let us take a system of co-ordinates in which the equations of Newtonian mechanics hold good. In order to render our presentation more precise and to distinguish this system of co-ordinates verbally from others which will be introduced hereafter, we call it the "stationary system."

If a material point is at rest relatively to this system of co-ordinates, its position can be defined relatively thereto by the employment of rigid standards of measurement and the methods of Euclidean geometry, and can be expressed in Cartesian co-ordinates.

If we wish to describe the motion of a material point, we give the values of its co-ordinates as functions of the time. Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by "time." We have to take into account that all our judgments in which time plays a part are always judgments of simultaneous events. If, for instance, I say, "That train arrives here at 7 o'clock," I mean something like this: "The pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events." It might appear possible to overcome all the difficulties attending the definition of "time" by substituting "the position of the small hand of my watch" for "time." And in fact such a definition is satisfactory when we are concerned with defining a time exclusively for the place where the watch is located; but it is no longer satisfactory when we have to connect in time series of events occurring at different places, or—what comes to the same thing—to evaluate the times of events occurring at places remote from the watch.

We might, of course, content ourselves with time values determined by an observer stationed together with the watch at the origin of the co-ordinates, and co-ordinating the corresponding positions of the hands with light signals, given out by every event to be timed, and reaching him through empty space. But this co-ordination has the disadvantage that it is not independent of the standpoint of the observer with the watch or clock, as we know from experience. We arrive at a much more practical determination along the following line of thought.

If at the point A of space there is a clock, an observer at A can determine the time values of events in the immediate proximity of A by finding the positions of the hands which are simultaneous with these events. If there is at the point B of space another clock in all respects resembling the one at A, it is possible for an observer at B to determine the time values of events in the immediate neighbourhood of B. But it is not possible without further assumption to compare, in respect of time, an event at A with an event at B. We have so far defined only an "A time" and a "B time." We have not defined a common "time" for A and B, for the latter cannot be defined at all unless we establish by definition that the "time" required by light to travel from A to B equals the "time" it requires to travel from B to A. Let a ray of light start at the "A time" t_A *from A towards B, let it at the "B time"* t_B *be reflected at B in the direction of A, and arrive again at A at the "A time"* t'_A*. In accordance with definition the two clocks synchronize if*

$$t_B - t_A = t'_A - t_B.$$

We assume that this definition of synchronism is free from contradictions, and possible for any number of points; and that the following relations are universally valid:

- 1) If the clock at B synchronizes with the clock at A, the clock at A synchronizes with the clock at B.

- 2) If the clock at A synchronizes with the clock at B and also with the clock at C, the clocks at B and C also synchronize with each other.

Thus with the help of certain imaginary physical experiments we have settled what is to be understood by synchronous stationary clocks located at different places, and have evidently obtained a definition of "simultaneous," or "synchronous," and of "time." The "time" of an event is that which is given simultaneously with the event by a stationary clock located at the place of the event, this clock being synchronous, and indeed synchronous for all time determinations, with a specified stationary clock.

In agreement with experience we further assume the quantity $\frac{2AB}{(t'_A - t_A)} = c$*, to be a universal constant—the velocity of light in empty space. It is essential to have time defined by means of stationary clocks in the stationary system, and the time now defined being appropriate to the stationary system we call it "the time of the stationary system."*

Section 3 (pp. 43-48)

Let us in "stationary" space take two systems of co-ordinates, i.e. two systems, each of three rigid material lines, perpendicular to one another, and issuing from a point. Let the axes of X of the two systems coincide, and their axes of Y and Z respectively be parallel. Let each system be provided with a rigid measuring-rod and a number of clocks, and let the two measuring-rods, and likewise all the clocks of the two systems, be in all respects alike. Now to the origin of one of the two systems (k) let a constant velocity v be imparted in the direction of the increasing x of the other stationary system (K), and let this velocity be communicated to the axes of the co-ordinates, the relevant measuring-rod, and the clocks. To any time of the stationary system K there then will correspond a definite position of the axes of the moving system, and from reasons of symmetry we are entitled to assume that the motion of k may be such that the axes of the moving system are at the time t (this "t" always denotes a time of the stationary system) parallel to the axes of the stationary system. We now imagine space to be measured from the stationary system K by means of the stationary measuring-rod, and also from the moving system k by means of the measuring-rod moving with it; and that we thus obtain the co-ordinates x, y, z, and ξ, η, ζ respectively. Further, let the time t of the stationary system be determined for all points thereof at which there are clocks by means of light signals in the manner indicated in § 1; similarly let the time τ of the moving system be determined for all points of the moving system at which there are clocks at rest relatively to that system by applying the method, given in § 1, of light signals between the points at which the latter clocks are located. To any system of values x, y, z, t, which completely defines the place and time of an event in the stationary system, there belongs a system of values ξ, η, ζ, τ , determining that event relatively to the system k, and our task is now to find the system of equations connecting these quantities. In the first place it is clear that the equations must be linear on account of the properties of homogeneity which we attribute to space and time. If we place x' = x − vt, it is clear that a point at rest in the system k must have a system of values x', y, z, independent of time. We first define τ as a function of x', y, z, and t. To do this we have to express in equations that τ is nothing else than the summary of the data of clocks at rest in system k, which have been synchronized according to the rule given in § 1. From the origin of system k let a ray be emitted at the time τ_0 *along the X-axis to x', and at the time* τ_1 *be reflected thence to the origin of the coordinates, arriving there at the time* τ_2*; we then must have*

$$\tfrac{1}{2}(\tau_0 + \tau_2) = \tau l,$$

or, by inserting the arguments of the function τ and applying the principle of the constancy of the velocity of light in the stationary system:

$$\frac{1}{2}\left[\tau(0,0,0,t) + \tau\left(0,0,0,t + \frac{x'}{(c-v)} + \frac{x'}{(c+v)}\right)\right] = \tau\left(x',0,0,t + \frac{x'}{(c-v)}\right)$$

Hence, if x ' be chosen infinitesimally small,

$$\frac{1}{2}\left(\frac{1}{(c-v)} + \frac{1}{(c+v)}\right)\frac{\partial\tau}{\partial x} = \frac{\partial\tau}{\partial x'} + \frac{1}{(c-v)}\frac{\partial\tau}{\partial t}$$

or

$$\frac{\partial\tau}{\partial x'} + \frac{v}{(c^2 - v^2)}\frac{\partial\tau}{\partial t} = 0.$$

It is to be noted that instead of the origin of the co-ordinates we might have chosen any other point for the point of origin of the ray, and the equation just obtained is therefore valid for all values of x', y, z. An analogous consideration—applied to the axes of Y and Z—it being borne in mind that light is always propagated along these axes, when viewed from the stationary system, with the velocity $\sqrt{(c^2 - v^2)}$ *gives us*

$$\frac{\partial\tau}{\partial y}=0,\frac{\partial\tau}{\partial z}=0.$$

Since τ is a linear function, it follows from these equations that

$$\tau=a\left(t-\frac{v}{(c^2-v^2)}x'\right)$$

where a is a function $\phi(v)$ at present unknown, and where for brevity it is assumed that at the origin of k, τ = 0, when t = 0. With the help of this result we easily determine the quantities ξ, η, ζ by expressing in equations that light (as required by the principle of the constancy of the velocity of light, in combination with the principle of relativity) is also propagated with velocity c when measured in the moving system. For a ray of light emitted at the time τ = 0 in the direction of the increasing ξ

$$\xi\ =\ c\tau \text{ or } \xi=ac\left(t-\frac{v}{(c^2-v^2)}x'\right).$$

But the ray moves relatively to the initial point of k, when measured in the stationary system, with the velocity c − v, so that x'/(c − v)= t. If we insert this value of t in the equation for ξ, we obtain

$$\xi=a\frac{c^2}{(c^2-v^2)}x'.$$

In an analogous manner we find, by considering rays moving along the two other axes, that $\eta\ =\ c\tau$ or $\eta=ac\left(t-\frac{v}{(c^2-v^2)}x'\right)$ when $\frac{y}{\sqrt{c^2-v^2}}=t$, x' = 0. Thus $\eta\ =\frac{ac}{\sqrt{c^2-v^2}}y$ and $\zeta\ =\frac{ac}{\sqrt{c^2-v^2}}z$. Substituting for x' its value, we obtain

$$\tau\ =\ \phi(v)\beta\left(t\ -\ v\frac{x}{c^2}\right),\xi\ =\ \phi(v)\beta(x\ -\ vt),\eta\ =\ \phi(v)y,\zeta\ =\ \phi(v)z$$

where

$$\beta=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

and ϕ is an as yet unknown function of v. If no assumption whatever be made as to the initial position of the moving system and as to the zero point of τ , an additive constant is to be placed on the right side of each of these equations.

We now have to prove that any ray of light, measured in the moving system, is propagated with the velocity c, if, as we have assumed, this is the case in the stationary system; for we have not as yet furnished the proof that the principle of the constancy of the velocity of light is compatible with the principle of relativity. At the time t = τ = 0, when the origin of the co-ordinates is common to the two systems, let a spherical wave be emitted therefrom, and be propagated with the velocity c in system K. If (x, y, z) be a point just attained by this wave, then $x^2+y^2+z^2=c^2t^2$. Transforming this equation with the aid of our equations of transformation we obtain after a simple calculation $\xi^2+\eta^2+\zeta^2=c^2\tau^2$. The wave under consideration is therefore no less a spherical wave with velocity of propagation c when viewed in the moving system. This shows that our two fundamental principles are compatible. In the equations of transformation which have been developed there enters an unknown function ϕ of v, which we will now determine. For this purpose we introduce a third system of co-ordinates K' , which relatively to the system k is in a state of parallel translatory motion parallel to the axis of Ξ,† such that the origin of co-ordinates of system K' moves with velocity −v on the axis of Ξ. At the time t = 0 let all three origins coincide, and when t = x = y = z = 0 let the time t ' of the system K' be zero. We call the co-ordinates, measured in the system K' , x' ,y' ,z', and by a twofold application of our equations of transformation we obtain

$$t'\ =\ \phi(-v)\beta(-v)\left(\tau\ +\ v\frac{\xi}{c^2}\right)\ =\ \phi(v)\phi(-v)t,$$

$$x'\ =\ \phi(-v)\beta(-v)(\xi\ +\ v\tau\ )\quad =\ \phi(v)\phi(-v)x,$$

$$y' = \phi(-v)\eta \qquad = \phi(v)\phi(-v)y,$$

$$z' = \phi(-v)\zeta \qquad = \phi(v)\phi(-v)z.$$

Since the relations between x', y', z' and x, y, z do not contain the time t, the systems K and K' are at rest with respect to one another, and it is clear that the transformation from K to K' must be the identical transformation. Thus $\phi(v)\phi(-v) = 1$.

We now inquire into the signification of $\phi(v)$*. We give our attention to that part of the axis of Y of system k which lies between* $\xi = 0, \eta = 0, \zeta = 0$ *and* $\xi = 0, \eta = l, \zeta = 0$*. This part of the axis of Y is a rod moving perpendicularly to its axis with velocity v relatively to system K. Its ends possess in K the co-ordinates*

$$x_1 = vt, \qquad y_1 = \frac{l}{\phi(-v)}, \qquad z_1 = 0$$

and

$$x_2 = vt, \qquad y_2 = 0, \qquad z_2 = 0.$$

The length of the rod measured in K is therefore $l/\phi(v)$*; and this gives us the meaning of the function* $\phi(v)$*. From reasons of symmetry it is now evident that the length of a given rod moving perpendicularly to its axis, measured in the stationary system, must depend only on the velocity and not on the direction and the sense of the motion. The length of the moving rod measured in the stationary system does not change, therefore, if v and* $-v$ *are interchanged. Hence follows that*

$$y_1 = \frac{l}{\phi(v)} = \frac{l}{\phi(-v)},$$

or

$$\phi(v) = \phi(-v).$$

It follows from this relation and the one previously found that $\phi(v) = 1$*, so that the transformation equations which have been found become*

$$\tau = \beta\left(t - \frac{vx}{c^2}\right),$$

$$\xi = \beta(x - vt),$$

$$\eta = y,$$

$$\zeta = z,$$

where

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Section 4 (pp. 48-50)

We envisage a rigid sphere of radius R, at rest relatively to the moving system k, and with its centre at the origin of co-ordinates of k. The equation of the surface of this sphere moving relatively to the system K with velocity v is $\xi^2 + \eta^2 + \zeta^2 = R^2$*. The equation of this surface expressed in x, y, z at the time t=0 is*

$$\frac{x^2}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}+y^2+z^2=R^2.$$

A rigid body which, measured in a state of rest, has the form of a sphere, therefore has in a state of motion—viewed from the stationary system—the form of an ellipsoid of revolution with the axes

$$R\sqrt{1-\frac{v^2}{c^2}}\,, R, R.$$

Thus, whereas the Y and Z dimensions of the sphere (and therefore of every rigid body of no matter what form) do not appear modified by the motion, the X dimension appears shortened in the ratio

1 : $\sqrt{1-\frac{v^2}{c^2}}$, *i.e. the greater the value of v, the greater the shortening. For v = c all moving objects—viewed from the "stationary" system—shrivel up into plane figures. For velocities greater than that of light our deliberations become meaningless; we shall, however, find in what follows, that the velocity of light in our theory plays the part, physically, of an infinitely great velocity. It is clear that the same results hold good for bodies at rest in the "stationary" system, viewed from a system in uniform motion. Further, we imagine one of the clocks which are qualified to mark the time t when at rest relatively to the stationary system, and the time τ when at rest relatively to the moving system, to be located at the origin of the co-ordinates of k, and so adjusted that it marks the time τ . What is the rate of this clock, when viewed from the stationary system? Between the quantities x, t, and τ , which refer to the position of the clock, we have, evidently, x = vt and*

$$\tau=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t-\frac{vx}{c^2}\right).$$

Therefore,

$$\tau=t\sqrt{1-\frac{v^2}{c^2}}=t-\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)t$$

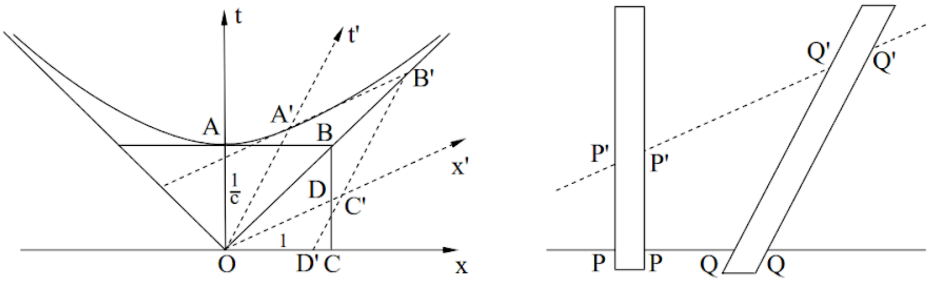
whence it follows that the time marked by the clock (viewed in the stationary system) is slow by $1-\sqrt{1-\frac{v^2}{c^2}}$ *seconds per second, or—neglecting magnitudes of fourth and higher order—by* $\frac{1}{2}\frac{v^2}{c^2}$. *From this there ensues the following peculiar consequence. If at the points A and B of K there are stationary clocks which, viewed in the stationary system, are synchronous; and if the clock at A is moved with the velocity v along the line AB to B, then on its arrival at B the two clocks no longer synchronize, but the clock moved from A to B lags behind the other which has remained at B by* $\frac{1}{2}t\frac{v^2}{c^2}$ *(up to magnitudes of fourth and higher order), t being the time occupied in the journey from A to B. It is at once apparent that this result still holds good if the clock moves from A to B in any polygonal line, and also when the points A and B coincide. If we assume that the result proved for a polygonal line is also valid for a continuously curved line, we arrive at this result: If one of two synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t seconds, then by the clock which has remained at rest the travelled clock on its arrival at A will be* $\frac{1}{2}t\frac{v^2}{c^2}$ *second slow. Thence we conclude that a balance-clock at the equator must go more slowly, by a very small amount, than a precisely similar clock situated at one of the poles under otherwise identical conditions.*

Minkowski excerpts (Minkowski, 1908. Translated by Lewertoff & Petkov, 2012)

Section 1 (pp. 39-43)

Gentlemen! The views of space and time which I want to present to you arose from the domain of experimental physics, and therein lies their strength. Their tendency is radical. From now onwards space by itself and time by itself will recede completely to become mere shadows and only a type of union of the two will still stand independently on its own. I want to show first how to move from the currently adopted mechanics through purely mathematical reasoning to modified ideas about space and time. The equations of Newtonian mechanics show a twofold invariance. First, their form is preserved when subjecting the specified spatial coordinate system to any change of position; second, when it changes its state of motion, namely when any uniform translation is impressed upon it; also, the zero point of time plays no role. When one feels ready for the axioms of mechanics, one is accustomed to regard the axioms of geometry as settled and probably for this reason those two invariances are rarely mentioned in the same breath. Each of them represents a certain group of transformations for the differential equations of mechanics. The existence of the first group can be seen as reflecting a fundamental characteristic of space. One always tends to treat the second group with disdain in order to unburden one's mind that one can never determine from physical phenomena whether space, which is assumed to be at rest, may not after all be in uniform translation. Thus these two groups lead completely separate lives side by side. Their entirely heterogeneous character may have discouraged any intention to compose them. But it is the composed complete group as a whole that gives us to think. We will attempt to visualize the situation graphically. Let x, y, z be orthogonal coordinates for space and let t denote time. The objects of our perception are always connected to places and times. No one has noticed a place other than at a time and a time other than at a place. However I still respect the dogma that space and time each have an independent meaning. I will call a point in space at a given time, i.e. a system of values x, y, z, t a worldpoint. The manifold of all possible systems of values x, y, z, t will be called the world. With a hardy piece of chalk I can draw four world axes on the blackboard. Even one drawn axis consists of nothing but vibrating molecules and also makes the journey with the Earth in the Universe, which already requires sufficient abstraction; the somewhat greater abstraction associated with the number 4 does not hurt the mathematician. To never let a yawning emptiness, let us imagine that everywhere and at any time something perceivable exists. In order not to say matter or electricity I will use the word substance for that thing. We focus our attention on the substantial point existing at the worldpoint x, y, z, t and imagine that we can recognize this substantial point at any other time. A time element dt may correspond to the changes dx, dy, dz *of the spatial coordinates of this substantial point. We then get an image, so to say, of the eternal course of life of the substantial point, a curve in the world, a worldline, whose points can be clearly related to the parameter t from* $-\infty$ *to* $+\infty$*. The whole world presents itself as resolved into such worldlines, and I want to say in advance, that in my understanding the laws of physics can find their most complete expression as interrelations between these worldlines. Through the concepts of space and time the x, y, z-manifold* $t = 0$ *and its two sides* $t > 0$ *and* $t < 0$ *fall apart. If for simplicity we hold the chosen origin of space and time fixed, then the first mentioned group of mechanics means that we can subject the x, y, z-axes at* $t = 0$ *to an arbitrary rotation about the origin corresponding to the homogeneous linear transformations of the expression* $x^2 + y^2 + z^2$*. The second group, however, indicates that, also without altering the expressions of the laws of mechanics, we may replace x, y, z, t by* $x - \alpha t, y - \beta t, z - \gamma t, t$*, where* α, β, γ *are any constants. The time axis can then be given a completely arbitrary direction in the upper half of the world* $t > 0$*. What has now the requirement of orthogonality in space to do with this complete freedom of choice of the direction of the time axis upwards? To establish the connection we take a positive parameter c and look at the structure* $c^2t^2 - x^2 - y^2 - z^2 = 1$.

Fig. 1



It consists of two sheets separated by t = 0 by analogy with a two-sheeted hyperboloid. We consider the sheet in the region t > 0 and we will now take those homogeneous linear transformations of x, y, z, t in four new variables x', y', z' ,t' so that the expression of this sheet in the new variables has the same form. Obviously, the rotations of space about the origin belong to these transformations. A full understanding of the rest of those transformations can be obtained by considering such among them for which y and z remain unchanged. We draw (Fig. 1) the intersection of that sheet with the plane of the x- and the t-axis, i.e. the upper branch of the hyperbola $c^2t^2 - x^2 = 1$ *with its asymptotes. Further we draw from the origin O an arbitrary radius vector OA' of this branch of the hyperbola; then we add the tangent to the hyperbola at A' to intersects the right asymptote at B' ; from OA'B' we complete the parallelogram OA'B'C' ; finally, as we will need it later, we extend B'C' so that it intersects the x-axis at D'. If we now regard OC' and OA' as axes for new coordinates x', t', with the scale units OC' = 1, OA' = 1/c, then that branch of the hyperbola again obtains the expression* $c^2t'^2 - x'^2 = 1$*, t' > 0, and the transition from x, y, z, t to x', y', z', t' is one of the transformations in question. These transformations plus the arbitrary displacements of the origin of space and time constitute a group of transformations which still depends on the parameter c and which I will call* G_C*. If we now increase c to infinity, so 1/c converges to zero, it is clear from the figure that the branch of the hyperbola leans more and more towards the x-axis, that the angle between the asymptotes becomes greater, and in the limit that special transformation converts to one where the t'-axis may be in any upward direction and x' approaches x ever more closely. By taking this into account it becomes clear that the group* G_C *in the limit* $c = \infty$*, that is the group* G_∞*, is exactly the complete group which is associated with the Newtonian mechanics. In this situation, and since* G_C *is mathematically more understandable than* G_∞*, there could have probably been a mathematician with a free imagination who could have come up with the idea that at the end natural phenomena do not actually possess an invariance with the group* G_∞*, but rather with a group* G_C *with a certain finite c, which is extremely great only in the ordinary units of measurement. Such an insight would have been an extraordinary triumph for pure mathematics. Now mathematics expressed only staircase wit here, but it has the satisfaction that, due to its happy antecedents with their senses sharpened by their free and penetrating imagination, it can grasp the profound consequences of such remodelling of our view of nature. I want to make it quite clear what the value of c will be with which we will be finally dealing. c is the velocity of the propagation of light in empty space. To speak neither of space nor of emptiness, we can identify this magnitude with the ratio of the electromagnetic to the electrostatic unit of the quantity of electricity. The existence of the invariance of the laws of nature with respect to the group* G_C *would now be stated as follows: From the entirety of natural phenomena, through successively enhanced approximations, it is possible to deduce more precisely a reference system x, y, z, t, space and time, by means of which these phenomena can be then represented according to certain laws. But this reference system is by no means unambiguously determined by the phenomena. One can still change the reference system according to the transformations of the above group* G_C *arbitrarily without changing the expression of the laws of nature in the process. For example, according to the figure depicted above one can call t' time, but then must necessarily, in connection with this, define space by the manifold of three parameters x', y, z in which the laws of physics would then have exactly the same expressions by means of x', y, z, t' as by means of x, y, z, t. Hereafter we would then have in the world no more the space, but an infinite number of spaces analogously as there is an infinite number of planes in three-dimensional space. Three-dimensional geometry becomes a chapter in four-dimensional physics. You see why I said at the beginning that space and time will recede completely to become mere shadows and only a world in itself will exist.*

Section 2 (pp. 43-45)

Now the question is, what circumstances force us to the changed view of space and time, does it actually never contradict the phenomena, and finally, does it provide advantages for the description of the phenomena? Before we discuss these questions, an important remark is necessary. Having individualized space and time in some way, a straight worldline parallel to the t-axis corresponds to a stationary substantial point, a straight line inclined to the t-axis corresponds to a uniformly moving substantial point, a somewhat curved worldline corresponds to a non-uniformly moving substantial point. If at any worldpoint x, y, z, t there is a worldline passing through it and we find it parallel to any radius vector OA' of the previously mentioned hyperboloidal sheet, we may introduce OA' as a new time axis, and with the thus given new concepts of space and time, the substance at the worldpoint in question appears to be at rest. We now want to introduce this fundamental axiom: With appropriate setting of space and time the substance existing at any worldpoint can always be regarded as being at rest. This axiom means that at every worldpoint the expression $c^2dt^2 - dx^2 - dy^2 - dz^2$ *is always positive, which is equivalent to saying that any velocity v is always smaller than c. Then c would be an upper limit for all substantial velocities and that is precisely the deeper meaning of the quantity c. In this understanding the axiom is at first glance slightly displeasing. It should be noted, however, that a modified mechanics, in which the square root of that second order differential expression enters, is now gaining ground, so that cases with superluminal velocity will play only such a role as that of figures with imaginary coordinates in geometry. The impulse and true motivation for accepting the group* G_C *came from noticing that the differential equation for the propagation of light waves in the empty space possesses that group* G_C*. On the other hand, the concept of a rigid body has meaning only in a mechanics with the group* G_∞*. If one has optics with* G_C*, and if, on the other hand, there were rigid bodies, it is easy to see that one t-direction would be distinguished by the two hyperboloidal sheets corresponding to* G_C *and* G_∞*, and would have the further consequence that one would be able, by using appropriate rigid optical instruments in the laboratory, to detect a change of phenomena at various orientations with respect to the direction of the Earth's motion. All efforts directed towards this goal, especially a famous interference experiment of Michelson had, however, a negative result. To obtain an explanation, H. A. Lorentz made a hypothesis, whose success lies precisely in the invariance of optics with respect to the group* G_C*. According to Lorentz every body moving at a velocity v must experience a reduction in the direction of its motion namely in the ratio 1 :* $\sqrt{1-\frac{v^2}{c^2}}$*. This hypothesis sounds extremely fantastical. Because the contraction is not to be thought of as a consequence of resistances in the ether, but merely as a gift from above, as an accompanying circumstance of the fact of motion. I now want to show on our figure that the Lorentzian hypothesis is completely equivalent to the new concept of space and time, which makes it much easier to understand. If for simplicity we ignore y and z and think of a world of one spatial dimension, then two strips, one upright parallel to the t-axis and the other inclined to the t-axis (see Fig. 1), are images for the progression in time of a body at rest and a body moving uniformly, where each preserves a constant spatial dimension. OA' is parallel to the second strip, so we can introduce t' as time and x' as a space coordinate and then it appears that the second body is at rest, whereas the first – in uniform motion. We now assume that the first body has length l when considered at rest, that is, the cross section PP of the first strip and the x-axis is equal to l · OC, where OC is the measuring unit on the x-axis, and, on the other hand, that the second body has the same length l when regarded at rest; then the latter means that the cross-section of the second strip measured parallel to the x'-axis is Q'Q' = l · OC' . We have now in these two bodies images of two equal Lorentz electrons, one stationary and one uniformly moving. But if we go back to the original coordinates x, t, we should take as the dimension of the second electron the cross section QQ of its associated strip parallel to the x-axis. Now as Q'Q' = l·OC' , it is obvious that QQ = l · OD' . If dx/dt for the second strip is = v, an easy calculation gives OD' = OC ·* $\sqrt{1-\frac{v^2}{c^2}}$*, therefore also P P : QQ = 1:* $\sqrt{1-\frac{v^2}{c^2}}$*. This is the meaning of the Lorentzian hypothesis of the contraction of electrons in motion. Regarding, on the other hand, the second electron as being at rest, that is, adopting the reference system x' , t', the length of the first electron will be the cross section P'P' of its strip parallel to OC' , and we would find the first electron shortened with respect to the second in exactly the same proportion; from the figure we also see that P'P' : Q'Q' = OD : OC' = OD' : OC = QQ : PP. Lorentz called t' , which is a combination of x and t, local time of the uniformly moving electron, and associated a physical construction with this concept for a better understanding of the contraction hypothesis. However, it is to the credit of A. Einstein who first realized clearly that the time of one of the electrons is as good as that of the other, i.e. that t and t' should be treated equally. With this, time was deposed from its status as a concept unambiguously determined by the phenomena. The concept of space was shaken neither by Einstein nor by Lorentz, maybe because in the abovementioned special transformation, where the plane of x' , t' coincides with the plane x, t, an interpretation is possible as if the x-axis of space preserved its position. To step over the concept of space in such a way is an instance of what can be achieved only due to the audacity of mathematical culture. After this further* *step, which is indispensable for the true understanding of the group G_C, I think the word relativity postulate used for the requirement of invariance under the group G_C is very feeble. Since the meaning of the postulate is that through the phenomena only the four-dimensional world in space and time is given, but the projection in space and in time can still be made with certain freedom, I want to give this affirmation rather the name the postulate of the absolute world (or shortly the world postulate).*

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