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Research shows the importance of the development of the learning and teaching of mathematical modelling in secondary school but also highlights some barriers, especially concerning the difficulties of teachers to implement modelling activities. An epistemological study of mathematical modelling and a contemporary epistemological study of modelling experts' practices led us to identify horizontal mathematization as a crucial component of mathematical modelling. Based on this conceptualization, we then came up with a task for teachers to implement in their classrooms, within a professional development program, which provides us some data to study the constraints and conditions of teaching practices concerning modelling. Our results suggest that a teaching strategy stressing horizontal mathematization is likely to foster a consistent modelling activity of students and that teachers don't identify horizontal mathematization as a learning issue.

Keywords: Horizontal mathematization, teaching practices, modelling.

Introduction

Research in mathematics education shows how developing the learning and teaching of mathematical modelling in secondary school is important, but also points out some hindrances, especially concerning the difficulties for teachers to implement modelling activities in their classrooms. Numerous studies point out some obstacles concerning teaching practices (Schmidt, 2011; Blum & Leiss, 2007; Kaiser et al., 2011; Maaß & Gurlitt, 2011; Ramirez, 2017; Barquero et al., 2018). For instance, mathematics teachers don't consider modelling as an essential component of mathematics learning; they doubt of their own competencies about mathematical modelling; they find it difficult to implement modelling tasks in class in particular because students might come up with a lot of different solutions and identifying what is at stake in such tasks is not easy; choosing an appropriate task is difficult for them. They also find it hard to identify students' difficulties and to evaluate modelling competencies; finally, they say they are lacking time to organize their teaching of modelling. Research on French teachers (Cabassut & Ferrando, 2017; Yvain-Prébiski, 2018) confirms that these international findings also apply to French teachers.

These hindrances mainly concern the conception and implementation of modelling activities in classrooms, which relate to elements formalised as institutional, cognitive, mediatory and personal constraints of teaching practices in the theoretical framework of the double approach of teaching practices of mathematics teachers (Robert & Rogalski, 2005; Vandebrouck, 2013). It leads us to hypothesize that overcoming some obstacles to the effective teaching of mathematical modelling in classrooms implies a more effective conceptualization of the modelling process. A literature review and a contemporary epistemological study of experts' practices in research about modelling in life sciences lead us to identify the crucial role of horizontal mathematization in this process and to characterize some forms of it that could enrich the conceptualization of mathematical modelling in classrooms. The hypotheses we want to test in our research are: first, a teaching strategy emphasizing horizontal mathematization is likely to foster a consistent modelling activity of students in classrooms; then, despite its identification by teachers as an important component of mathematical modelling, they don't seem to consider horizontal mathematization as a learning issue. In the next section, firstly we present our conceptualization of mathematical modelling, stressing horizontal mathematization as a crucial component of the process. In the second part, we are exposing the theoretical framework we used to study teaching practices and our experimental design. In a third part, we will expose our findings. Conclusion will allow us to design some perspectives about teacher training.

Horizontal mathematization: a fundamental component of mathematical modelling

Mathematics education researchers working on mathematical modelling in an educational perspective, especially in the RME framework (for example, Rasmussen et al. 2005; Barnes, 2005) consider the distinction introduced by Treffers (1978) and Freudenthal (1991), between horizontal mathematization which "leads from the world of life to the world of symbols" and vertical mathematization, as work within the mathematical system itself. In our research, we focused on the teaching and learning of mathematical modelling based on extra-mathematical situations. We then followed the idea of Israel (1996), who defines a mathematical model as "a piece of mathematics applied to a piece of reality" and defined different forms of horizontal mathematization that seem relevant to explore educational issues: choosing a piece of reality to question in order to answer the problem; identify and choose the relevant aspects of the piece of reality (context elements, attributes); relating the chosen aspects together in order to construct a mathematical model; and last, quantification (Chabot & Roux, 2011), which refers to the association of some aspects of reality to quantities (which essentially consists in measuring).

To deepen our epistemological study of modelling, we analysed the practices of experts (researchers) on mathematical modelling in life sciences, through interviews (Yvain, 2017). The main findings consist in the identification of three invariant features in the practices of researchers that contribute to the transformation of reality to mathematical solvable problems. The first invariant, which we labelled P_{i1}, consists in simplifying the problem and selecting a piece of reality. It supposes to identify relevant variables and choose relevant relations between the selected variables. The second invariant P_{i2} consists in choosing a model among those known by the researcher in order to initiate vertical mathematization, at the risk of further having to refine or reject the initial model. The third invariant consists in quantifying in order to compare the "real data" with the results obtained within the model. We assume that taking into account horizontal mathematization in the teaching and learning of modelling is likely to foster students' competencies.

Theoretical framework for the study of the teaching and learning process

Our conception of learning relies on the idea that students' learning depends on their activity which results from the tasks they have to solve in class. However, we assume that solving a task is not enough to guarantee conceptualising. Indeed, the role of the teacher is also to allow students to identify mathematical knowledge in their activity to ensure the "transformation of activity into learning". We will then consider that actively practicing horizontal mathematization when solving the task would constitute a first step on the way to the appropriation by students of horizontal mathematization within the process of modelling: it corresponds to a "local devolution" (Perrin-Glorian, 1997) of this learning issue. Considering that the devolution of it is complete (and hence that substantial opportunity for conceptualising is offered) supposes that the knowledge at stake is explicitly identified – by the teacher – as something to be learned (i.e. remembered in order to use it again later).

Our theoretical framework to study teaching practices is the double approach of mathematics teachers' practices, developed by Robert and Rogalski and their followers (Robert & Rogalski, 2005; Vandebrouck, 2013). Teaching practices are considered as complex, embedding five components. The first two of these are the cognitive and mediatory ones, which are directly observable in classrooms. The cognitive one corresponds to the teacher's choices regarding content and tasks, including their organization. Choices corresponding to class events and to the effective implementation in classrooms of the tasks make up the mediatory component. The three other ones are the social, institutional and personal components, which allow taking into account elements that crucially impact some choices made by teachers. This framework leads us to investigate constraints that are more likely to impact teachers' choices, would it be related to personal, social or institutional constraints. First, institutional constraints are essentially related to official instructions. Then, we have conducted analyses of classroom sessions, considering both the cognitive and the mediatory components of practices, and analyses of questionnaires filled by teachers during the development program, which allowed us to make some hypotheses on personal and social constraints.

Experimental design

Relying on our epistemological study, we tried to characterize tasks likely to support the learning of horizontal mathematization and hence students' activities inspired by P_{i1}-P_{i2} and P_{i3} practices. Such tasks should then be "realistic fictions" conceived as adaptations of a professional modelling problem. Here is an example of such a task that we designed for our experiments:

The tree: Botanists from Botanical Garden have discovered an exotic tree. To study this new species, the botanists have sketched the tree every year since 2013.

The botanists want to build a greenhouse to protect it. They believe it will have reached its full size by 2023. To help them, predict how the tree will be in 2023.

Figure 1: Text of the task "The tree"

In order to have teachers implementing this task in their classrooms, we worked with a group of teachers in the context of a professional development program about mathematical modelling[1]. Not only shall the program include the fact of asking teachers to implement the task in their classroom, but it will also do it with a specific scenario. The scenario included five sessions. Groups of three classes were conformed and all the classes interacted using a platform regulated by the designers of the program. We will not give more details about the professional development program (see Yvain & Modeste, 2018, for more details) except for one point which we will linger on in this paper: the original first phase of "questions-and-answers". The first step of the scenario requires teachers to devote the first session to students asking questions about the problem, to send them to two other classes and then to devote the second session to trying to answer those questions.

Collected data and methodology

In order to identify constraints related to the institutional component of teaching practices, a preliminary study of French official instructions for secondary school allowed us to identify the importance of mathematical modelling as a learning issue, but it remains a lot of implicitness about the role of horizontal mathematization in this process (Yvain, 2018). To explore the cognitive and mediatory components of teaching practices, we collected videos of the sessions in the classrooms of three teachers who took part to the program. We postulate that if teachers identify the learning issue related to horizontal mathematization, we would find traces of it in the cognitive and mediatory components of their practices, in the way they introduce the questions-and-answers phase to students and in the content they summarize at the end as what should students remember. We then designed four levels that indicate how much this learning issue is identified by teachers, according to what they had said during the introduction phase and the closing phase. These indicators are based on the forms of horizontal mathematization described above.

Identification of horizontal mathematization as a learning issue	In the introduction phase	In the closing phase
Level 1: no identification	No element indicating the function of the questions-and-answers phase in the solving of the problem	The teacher doesn't make a summary of what should be remembered or the summary doesn't mention anything related to horizontal mathematization.
Level 2: vague identification	Some elements indicating the function of the questions-and-answers phase in the solving of the problem	The summary includes the idea that some information was missing in the initial problem to be able to solve it.

[1] This program has been existing for ten years and was designed by a group where researchers and some teachers work collaboratively (one of the two authors of this paper is part of the group). Every schoolyear, this program offers a collaborative project to volunteering teachers with their classes from 6th grade till the end of secondary school.

Level 3: partial identification	Elements inducing the idea of the necessity of making choices	The summary mentions the necessity of making choices in order to be able to solve the problem.
Level 4: complete identification	Elements inducing the idea of the necessity of making choices on context features and about relevant attributes to model the problem. Idea that modelling the problem will depend on the choices made.	The summary mentions the necessity of making choices in order to be able to solve the problem and explains that different models could be used to solve the problem. The summary mentions the importance of choosing relevant attributes, depending on the problem to be solved, in order to choose a model.

Table 1: Indicators of the identification by teachers of horizontal mathematization as a learning issue

Findings: A task and a strategy effectively supporting students' activity of horizontal mathematization

Findings concerning students' activity during the phase of questions-and-answers

The analysis of questions and answers produced by students shows that they made simplifying hypotheses to handle the problem by selecting one or some pieces of reality: they focused on branches, the greenhouse, the leaves etc. They identified relevant variables that impact the real situation (for example, the number of branches) and non-relevant variables or elements of context (possibility to use organic fertilizer, or leaves); they chose relevant relations between the selected variables and used several mathematical tools (functions, proportionality, and geometry). These activities attest that students implemented P_{i1}–like activities.

Students also identified the necessity of modelling the situation to be able to predict the tree's growth. Students authentically questioned the variables to choose in order to solve the problem mathematically. The realistic aspect of the situation lead them to reflect on the important elements of context to be taken into account. This also helped them understand that not all information on the real context was useful to solve the problem or that it was suitable to ignore them in order not to come up with a too complex model. Our analysis showed that students mainly tried to use models that they knew (particularly, those using proportionality) and that they sometimes rejected it when they confronted the results they obtained to contingency, which attests their ability to implement P_{i2}-like activities.

Finally, the mention of the scale in the text of the task allowed students to use instruments to make measures, which lead them to quantify some attributes and confront measures with real data, which attests of P_{i3}-like activities. The analysis of traces of students' activities allows us to conclude that the characteristics of the task and the questions-and-answers phase fostered activities of students that resemble experts' practices and correspond to horizontal mathematization. This attests that the first level of devolution of horizontal mathematization was reached. However, what it still remains open for future research is how students responsibility was really involved in these activities and to what extent this issue was identified by teachers in a way that allows a full devolution of the learning issue.

Analyses of teachers' practices

The analyses of the classroom sessions we observed, with the study of answers teachers provided to a questionnaire during the development program, show that the impact on the motivation of students. The main advantages that teachers identify are the collaborative aspect of the questions-and-answers phase and the fact that this phase develops a better appropriation of the problem by students. About the mediatory component of teaching practices, the analysis of the different implementations show that teachers encourage students to produce questions and then answers in order to solve the problem. This is what guarantees the local "devolution" of horizontal mathematization to students, during the questions-and-answers phase. However, when considering teachers discourse during the introductory phase, it appears that the purpose of the questions-and-answers phase is not completely explicit. Two out of the three teachers discourses correspond to level 2 (see the first excerpt below) and one to level 3 (see the second excerpt below). Note that none of the teachers we observed attained level 4, neither for the introductory phase nor the conclusion phase.

Teacher 1	Your job is to come up with questions, things that you didn't understand, things that are missing, ok, here we miss some information, we need to know this or that to try to answer, or to clarify some things, you decide.
Teacher 2	The challenge of the session is to try to provide some elements of answer. Do you have an opinion, an answer to propose? What is your point of view? Ok and if you had to make choices, which choices would you make? Because in my opinion, some choices are necessary or they will never be able to do anything.

During the concluding phase, two teachers discourses correspond to level 2 ("the aim was to focus all the classes on the same problem in order to have everybody researching the same thing and to give you more information to proceed."); one teacher's discourse corresponds to level 3: "modelling is what we have been doing all along: making choices, ok? " All in all, even if some teachers mention the necessity of making choices to solve the problem, they don't seem to identify this necessity as a learning issue. In particular, they never offer students any formalization of the knowledge at stake about horizontal mathematization. Based on our theoretical framework, this makes us conclude that the devolution of the learning issue constituted by horizontal mathematization is not complete.

Discussion and conclusion

Our research shows that a conceptualization of mathematical modelling stressing the process of horizontal mathematization allows us to deepen our understanding of the learning and teaching issues related to mathematical modelling. Such conceptualization provides tools to support building efficient teaching strategies in order to guarantee consistent students' activities of horizontal mathematization. These strategies include, in particular, a questions-and-answers phase that seems to play a fundamental role. Let us note that these conclusions meet the preoccupations of other researchers working on the same subject. For example, the MERIA project also stresses the opportunities offered by a questioning phase in order to develop some skills about modelling.

However, our study also suggests that teachers, even if they identify the necessity of making choices in mathematical modelling, don't completely understand it as a learning issue. Yet, this appears as a condition for the complete devolution of the issue, in order to have students practice horizontal mathematization in ordinary classes, i.e. when teachers are not engaged in specific programs, imposing them teaching strategies. It leads us to postulate that a conceptualization of mathematical modelling in a way that includes horizontal mathematization, as a crucial and explicit component, would also constitute a good tool for teachers in order to overstep some of their difficulties and to improve their teaching skills about mathematical modelling – and hence, to improve learning. Our perspective is then to enrich our professional development program with an explicit work on horizontal mathematization with teachers and to make them able to identify it as a learning issue and as a tool to support the full devolution of it to students. For example, we suppose it would be fruitful to work with teachers on building summaries of such classroom sessions, suggesting them that elements about the necessity of making choices on relevant attributes and context elements in order to be able to model a situation and solve a problem would be valuable to explicit with students. Our research constitutes only a first step but points out a potential fruitful approach of teacher training about mathematical modelling, provided that teaching practices are considered in their complexity.

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