Design of situations fostering horizontal mathematization: benefits from epistemological analysis of experts' practices

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Abstract. This paper questions the teaching and learning of mathematical modeling. We develop an epistemological study of the mathematization in the practices of experts in two fields: life sciences and industrial sciences. We use the results of this contemporary epistemological study to analyze and support the relevance of some problems designed to foster the devolution of the mathematical modeling process to the students. Then, we present our choices for implementing these problems in the classroom in link with this objective, and compare the students' productions with the results of the epistemological analysis.

Résumé : Ce document questionne l'enseignement et l'apprentissage de la modélisation mathématique. Nous développons une étude épistémologique de la mathématisation dans les pratiques des experts dans deux domaines : les sciences de la vie et les sciences industrielles. Nous utilisons les résultats de cette étude épistémologique contemporaine pour analyser et étayer la pertinence de certains problèmes destinés à favoriser la dévolution du processus de modélisation mathématique aux étudiants. Ensuite, nous présentons nos choix pour la mise en œuvre de ces problèmes en classe en lien avec cet objectif, et nous comparons les productions des élèves avec les résultats de l'analyse épistémologique.

1. Introduction: contemporary epistemology on order to foster modeling activities for students

Our research interests are about mathematical modeling in mathematics problem solving in the classroom. Our approach is to investigate experts' practices as an epistemological reference to enlighten the teaching and the learning of mathematical modeling.
More specifically, in this paper, we investigate the practices mobilized during mathematical modeling in different contexts (here, the context of life sciences and the industrial problems). To do this, we will distinguish horizontal mathematization and vertical mathematization and take into account the dialectical relationship between them.
This will allow us to look into expert practices of modeling and contrast them with the mathematical activity in classroom in situations needing a modeling step from extra-mathematical situations.

1.1. Aim and research questions

Our research problem is "How can studying expert practices of modeling support the design and the implementation of modeling activities in classrooms?"

This communication relies on the distinction between horizontal mathematization and vertical mathematization (Treffers, 1978) introduced in the theoretical framework known currently as Realistic Mathematics Education (Freudenthal, 1991): the horizontal mathematization corresponds with the mathematical modeling that goes from the real situation to the mathematical world, whereas the vertical mathematization corresponds with the mathematical addressing of a problem.

> Horizontal mathematization leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly: this is vertical mathematization. The world of life is what is experienced as reality (in the sense I used the word before), as is a symbol world with regard to abstraction. (Freudenthal, 1991, p.41-42).

We cross this distinction with the epistemological work of Israël (1996) who considers that a mathematical model is "A piece of mathematics applied to a piece of reality" (Op.cit. p.11). He adds that "A single model not only describes different real situations, but this same piece of reality can also be represented by different models" (Ibidem). This led Yvain-Prébiski (2018) to define horizontal mathematization as choosing of a piece of reality, and then identifying and selecting some aspects of that piece of reality that can be mathematically addressed, in order to connect them to build a mathematical model.
To define vertical mathematization, we follow Treffers (1978): Vertical mathematization corresponds with the mathematical work inside the world of symbols, namely the mathematical treatment of a mathematical problem.
The figure 1 sketches the modeling process described above and highlights the dialectic relation between the horizontal and vertical mathematizations.

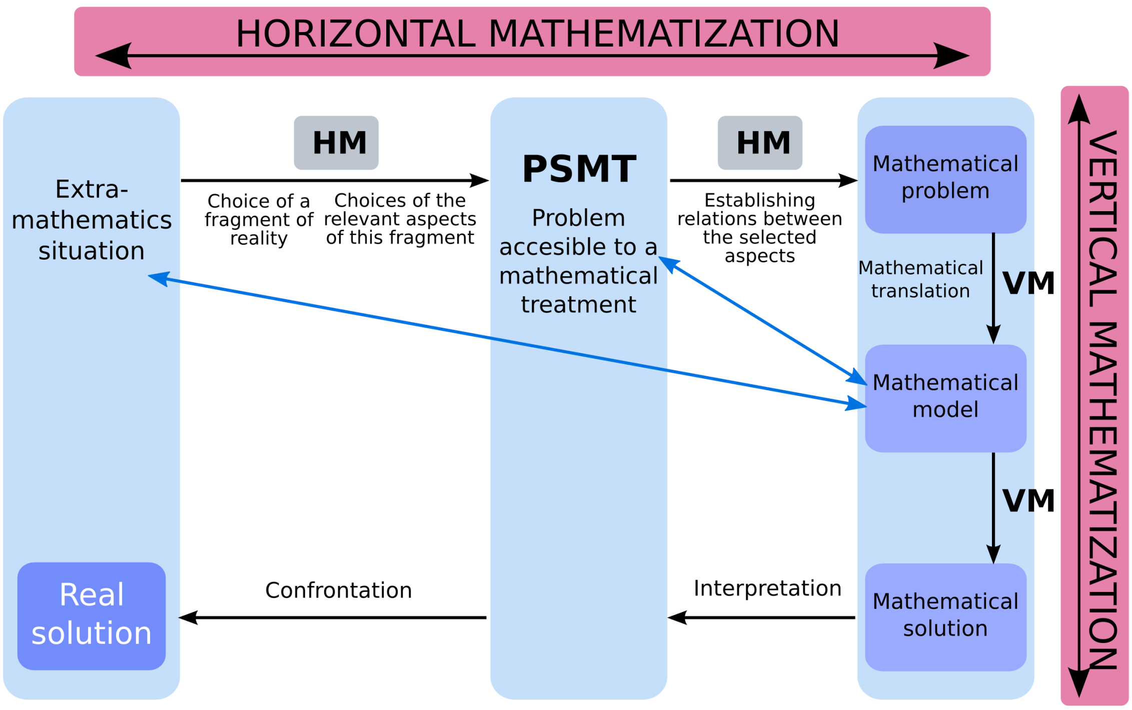

Figure 1. Modeling process, from Yvain-Prébiski (2018).

In such a context, our research problem can be rephrased into two research questions:
Q1. In expert practices of mathematical modeling, what are the invariant practices of horizontal mathematization and how do they interplay with context-specific practices?
Q2. How to take into account the results of the epistemological study to elaborate situations that foster the devolution of horizontal mathematization in the classroom, and to analyze students'

activity in such situations?

1.2. Methodology

To tackle these questions, our methodology is the following:

- Demarcate specific topics or areas where mathematical modeling is involved, but different enough from each other to permit the identification of both invariant practices and context-specific practices. We have selected life sciences and industrial problems as two contexts fitting with these criteria.
- Develop contemporary epistemological studies on modeling, by interviewing experts about their own modeling practices (guiding them to make explicit these practices); and identify invariant and context-specific practices.
- Design problems, trial them and analyze the students' activities through the lens of the epistemological study developed previously.

1.3. Area selected

Following the work of Yvain-Prébiski (2018) on modeling practices of experts in life sciences, we will conduct interviews with experts who use mathematical modeling in industrial sciences, with two purposes:

6. Identify invariants in their practices and compare them with those identified in life sciences,
7. Identify modeling practices that could be context-specific.

This second purpose will lead us to analyze again and reinterpret the interviews done by Yvain-Prébiski, to consider actions specific to the two chosen contexts. This second part of the work is ongoing and the first results will be presented here.

2. Contemporary epistemological study

2.1. Contemporary epistemological study: interviewing experts

In order to investigate experts' practices, we have decided, based on previous work of Yvain-Prebiski (2018), to interview experts in life sciences and industrial sciences on their modeling practices. To do this, we have used a *clarification interview* technique, which considers the interviewee as an informant, and consists in making this informant explain and detail a project of his where mathematics intervened. The aim is not to induce ideas from the interviewed expert but to bring the informer to make as explicit as possible his practice. The words *modeling* or *mathematizing* were only mentioned if the informant had mentioned them first.
The following interview grid has evolved over time (taking into account the difficulties or misunderstandings of previous interviews, but also introducing new questions or directions, based on the missing information or unexpected answers from previously interviewed informants).
The interview grid consists of three lines:

Line 1: information on the interviewee's education and career path of the informer and training in mathematics.

- Line 2: Present and make explicit a research project or an engineering project involving mathematics.
- Line 3: Define "modeling process"

This leads to the grid below:

Line 1:

What is your education path?
What is your career path?
What place do mathematics and computer science have in your training?

Line 2:

Can you describe, from the beginning to the end, related to life science/industrial problems where mathematics have been involved?

Relaunch questions:

- How did the project get started? What was the initial question?
- Which (mathematical, computational, experimental) choices have been made to start, & to go on?
- How did you choose the mathematical and computational tools and frameworks?
- To which extent have mathematics (and computer science) been essential in the project? How much have they been involved in the project?
- Has it been necessary to modify or abandon some choices?
- Are there other disciplines involved in your work? In the project?
- Did you collaborate with mathematicians or computer scientists?
- Have you needed to reformulate or rebuild your problem or your questions?
- How has the original question evolved?
- How do you validate the results?
- How do you test the validity of your models? (or mathematical/computational choices?)
- When do you reject a model? (or a mathematical/computational choice?)
- What kind of difficulty have your encountered in the project?

Line 3:

Can you formulate what is, for you, a "modeling process"?
Do you consider your own practice/definition as representative of or shared in your community?

Yvain-Prebiski has interviewed 5 experts in life sciences and the authors have interviewed (until now) 3 experts in industrial sciences (1 in logistics, 2 in operations research). We present below the results of this ongoing work.

2.2. Context of life sciences

In her PhD thesis, Yvain-Prébiski (2018) led such a contemporary epistemological study on the practices of researchers using mathematical modeling in life sciences. The main findings consist in the identification of six invariant features in the practices of researchers which contribute to the transformation of reality to mathematically solvable problems:

- simplifying the problem and selecting a piece of reality; it is expected to identify relevant variables and choose relevant relations between the selected variables.
- choosing a model among those known by the researcher in order to initiate vertical mathematization, at the risk of having to refine or reject the initial model later.
- quantifying in order to compare the "real data" with the results obtained within the model.
- using computer simulation to obtain results with the chosen model. Often it is with computer simulation that the researcher will test whether his choice of the fragment of reality associated with the model he induces is consistent with the observations resulting from the experiment or simulated experiment.
- ideally, hoping to obtain a result that is becoming more widespread and that would then become a new model itself.

- choosing a model by anticipating the feasibility of conducting an experiment that would allow the results obtained by the model to be compared with those obtained experimentally. There are real back and forth actions between the fragment of reality, the envisaged model, the first results obtained with the models, and the experimental data. This practice is part of the validation of the choice of model by researchers.

This contemporary epistemological study helped to better identify the form and the role of horizontal mathematization in a mathematical activity involving modeling.
Our objective is to extend this previous work to the context of industrial sciences, by interviewing researchers and professionals of this field. To do this, we will adapt the methodology used previously for life sciences, and compare and contrast results between the two application fields.

2.3. Context of industrial problems and preliminary conclusions

The six invariant features presented below also appear in the case of industrial problems, in particular the first three. The first results concerning industrial sciences allowed us to make some hypotheses about what might differ from life sciences:

- Human aspects in to be taken into account: In an industrial process, human operators can take action. This human factor needs to be incorporated into the model, and some theoretical solutions, even optimal, must be rejected if they are not understandable or acceptable for these operators. Thus the nature of the model and its validation can be different.
- Some relevant variables decided in advance: the manufacturer's problem generally comes with some information regarding the "variables" that can be "touched" and not "not touched". In this way, the experts deal with a problem which has been partially modeled beforehand.
- A lot of "fitting". Experts in industrial sciences insist on the strong part of the work making back-and-forth actions between the model and the problem, in order to adjust the model (make it "fit" the situation). This is allowed by the amount of data generally collected by the manufacturer.
- Status of simulations: It seems that simulations have a specific status in industrial sciences, according to the interviewed experts. This point has to be deepened, taking more into account the role of computation in the design of simulation, and the relation between models and simulations.
- Experimentation versus data: the experts in industrial sciences emphasize the role of data in the model design, which is more common than experiments (certainly for their cost). So, it is often the collected data that drive the model (or that is used as model). This is linked with point 3 and the following discussion.

Concerning the third and fifth points ("fitting" and data), the comparison with life sciences made us aware that this is also present in life sciences. Indeed, when a model is selected, a lot of work is done in both cases to make it fit the experimental data. But it seems that there is still a difference in industrial sciences, namely that the data do not come from experiments but from collection along the industrial process at stake, and that the aims of experts in industrial sciences are less the understanding of the phenomena than the optimization of the process.
This point must be deepened as this issue could differ between researchers' and engineers' points of view, more than between disciplines.
To conclude, this is an ongoing epistemological study, but we make the hypothesis that there are common principles between the processes of modeling in life sciences and industrial sciences (and, probably whatever domain of application).
But we have also begun to show some differences that are specific to life sciences and industrial

sciences. It would be important, from a didactical point of view, to be able to demarcate what is generic in modeling from what is domain-specific.

3. Design and implementation of problems for the classroom

3.1. Characteristics desirable for the implemented situations

Relying on our epistemological study, we intend to characterize problems likely to support the learning of horizontal mathematization and hence foster students' activities inspired by the invariant practices identified above. Such tasks should then be "realistic fictions" conceived (in English) as Adaptations of a Professional Modeling Problem (FRAPPM) and must meet the following criteria as much as possible:

- To bring students to reflect on the system they should model.
- To bring students to become conscious of:
  - the necessity to develop a model to solve a problem
  - the necessity to make choices to mathematically address the problem
  - the importance of the question set to them during the development of the model
  - the work behind the development of the model requires mathematical work within the model chosen to answer the questions.

The design of such situations has been developed within the RESCO (collaborative problem solving) group of the IREM of Montpellier (Research Institute for Teaching of Mathematics). This program has been existing for more than ten years and was designed by a group where researchers and teachers work collaboratively (the two authors of this paper are part of the group).
Based on these criteria, four FRAPPM have been developed last years:
In the context of life sciences:

- "The tree" (How to foresee the growth of a plant?)
- "Endangered species?" (How to forecast the evolution of animal population?)

Or in the context of industrial problems:

- "The warehouse" (How to optimize the location of a supply storage?)
- "The windows" (How to optimize the cutting of materials according to orders?)

The two first examples, trialed in 2016 and 2017, have been presented in the CIEAEM 69 (Yvain & Modeste, 2018, Yvain, 2018). In this paper, we will give details regarding the two last ones, trialed in 2018 and 2019 respectively.

3.2. Context of implementation

The implementation in classrooms of secondary education is coordinated through the ResCo program. The collaborative problem-solving device is based on exchanges between classes, working in groups of three, on the same research problem. All secondary levels from the 6th to the 12th grade are potentially concerned, adding a constraint on the choice of the problem to be proposed. In addition, this device is spread over five weeks (one session per week), during which collaboration between classes is organized. This implies that the problem proposed by the ResCo group should not be found on an Internet search engine, and, leading the group to formulate it in a new form.
A particularity of this program is its question and answer session designed to approach the problem more efficiently from the beginning. The first session proposed in the ResCo device aims to introduce students to realistic fiction and to have them formulate the questions they ask themselves

during their first research. The aim is to have the students ask themselves questions about the different possible choices that would allow them to deal with the problem mathematically. The questions developed by the students are addressed to the two classes with which their class is associated, by the teacher, via the ResCo forum.
During the second session, students receive questions from the other classes in their group. It is during the phase of the elaboration of the answers by the students that, on the one hand, the relevant questions for solving the problem will emerge and, on the other hand, different possible modeling choices will appear. During this session dedicated to the answers, the questions received lead to discussions that allow students to become aware of the need to make choices to deal mathematically with the problem, particularly around the identification of relevant quantities.
In the third week, the students discover the answers of the other classes and discuss these answers. Between the second and third weeks, on the basis of the questions and answers submitted to the forum, the ResCo group develops a "relaunched realistic fiction". It is addressed to all classes during this third week, in order to set common modeling choices to allow further collaboration in solving the same mathematical problem in every class. The intentions of the ResCo group are to make visible to students the need to make choices to solve the problem. During the fourth session, the students continue the research of this same mathematical problem, resulting from the modeling choices set by the "relaunched realistic fiction" of the ResCo team. During the last week, teachers are invited to carry out an assessment with their students to close the session. The ResCo group uses all the student productions posted on the forum to produce an assessment of the mathematical concepts and skills that the problem has enabled to be implemented, an assessment of the heuristic skills developed and the elements of a mathematical solution to the problem.
For more details about the professional development program or the scenario, see Yvain & Modeste (2018) and the workshop of Lavolé, Modeste, & Yvain-Prébiski presented at CIEAEM 71.

3.3. Detailed presentation of two of the implemented problems

Based on epistemological considerations on modeling and on the above characteristics and context, we have developed situations that aim to foster students' modeling activity. We describe here two of these situations (figures 2 and 3).

The Warehouse
A company has many factories which must be supplied weekly.
The map indicates:

- the position of the factories,
- the names given to the factories,
- the number of units of goods corresponding to the supplies needed by the factory every week.

The company want to build a warehouse from which the supplies will be delivered by truck to the factories. The maximal capacity of a truck is 120 units of goods.
The company wants to situate the warehouse the most economical way as possible. Can you help to decide where to settle the warehouse?

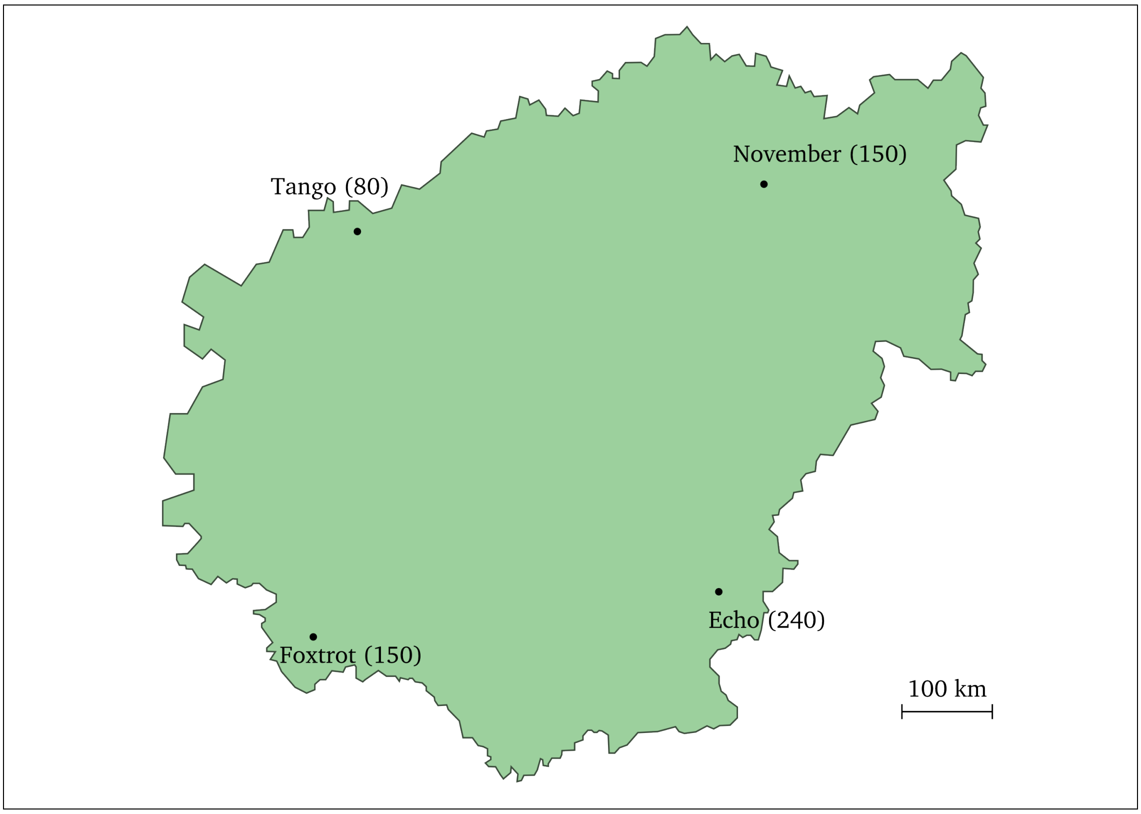

Figure 2. The situation called “The Warehouse” (experimented in 2018).

The windows panes
A company cuts rectangular window panes of four different dimensions:
210 cm x 215 cm ; 100 cm x 215 cm ; 100 cm x 125 cm ; 60 cm x 215 cm.
These window panes are cut out of large rectangular glass slabs measuring 600 cm x 320 cm.
The company is looking for a method in order to realize the cuts according to the orders while limiting the scrap.
To help the company, can you suggest a method realizing the cuts and minimizing the loss?

Figure 3. The situation called “The windows panes” (experimented in 2019).

These two problems fit with the characteristics of FRAPPM, stated above. In particular:

- they have been conceived as a transposition of modeling issues coming from professional scientific practices (work of the modeler): the first one comes from a common issue in logistics, and the second from a famous issue of optimization of a production process.
- their didactical variables (Brousseau & Warfield, 2014) are chosen in order to foster entry into horizontal mathematization:
  - In the first situation, the map, the type of journeys, the type of transportation and storage, the possibles interpretations of “the most economical way”, the needed approximations, the possible choices of the cost function, are decided for this purpose.
  - In the second situation, the nature of the cuts, the nature and organization of the orders, the allowed configurations of window panes in the glass slabs, the possible interpretations of “minimizing the loss”, the possible loss functions, are decided in order to foster the horizontal mathematization work.

In both cases, the modeling is supported by an optimization issue. It is important that the proposed situations need to be modeled for a specific purpose, and not for modeling itself. Here, it is optimizing; in the two other situations, it was making a prediction.
The careful design and analysis of the problems and their wording permits anticipation of the various mathematical problems that can arise from them and, based on the questions and answers sent by the classes and some other constraints, permits the design of a "relaunched realistic fiction" which is relevant and appropriate.

3.4. Analysis of the students' work

In previous work, Yvain-Prébiski (2018) developed and analyzed problems implemented in classrooms that, on the one hand, promotes the devolution of horizontal mathematization to students in their work, and, on the other hand, studies the possibles traces of transposition into classrooms of identified invariant practices. In order to analyze student's questions, she has defined three indicators of the devolution of horizontal mathematization to students:
The students' questions show them:

- building a model in order to respond to the given situation
- identifying the relevant features for mathematical processing
- seeing the relevance of the contextual elements to take into account to make the situation accessible to a mathematical treatment.

For the analysis of the student's answers, she has defined five indicators as follows:
The students' answers show

- the development of a model that enables a response to the situation
- choices of relevant quantities for mathematical processing
- choices of contextual elements
- analysis by the students of the relevance of a question regarding the situation
- the first mathematical work to answer the question.

Table 1 and 2 show examples of the students' work based on the "Warehouse" problem.

Table 1. Examples of students' work based on the "Warehouse" problem during the question phase.

Building a model in order to respond to the given situation	**Identifying the relevant features for mathematical processing**	**Questioning the relevance of the contextual elements to make the situation accessible to a mathematical treatment**
- Is it possible to supply plants beyond their demand? - Is the truck allowed to bring the goods to two factories in a row without returning to the warehouse? - Can central symmetry or axial symmetry be used? - Should the warehouse be placed closer to those who need more goods?	- What does "the most economical" mean?: to save gas or save money? - Do we want the truck to travel as few kilometers as possible? - Is the distance on the map important, the scale? - Are the numbers of goods requested by the plants accurate?	- What is the type of road between the warehouse and each plant? - What are the types of goods? - In which country or city are the different factories located? - What are the geographical constraints?

Table 2. Examples of students' work based on the "Warehouse" problem during the answers phase.

The development of a model that enables a response to the situation	**Choices of relevant quantities for mathematical processing**	**Choices of contextual elements**	**Analysis by the students of the relevance of a question regarding the situation**
Q: Should the truck's route be planned? It is necessary to plan the route of the trucks in order to optimize the organization. You have to find the most fuel-efficient route.	*Q: What types of roads should I take?* The fastest!	*Q: Are there any locations where it is not possible to place this warehouse?* For us, we have chosen to say no. But it's true that we wonder what happens when there's a lake.	*Q: What is the size of the warehouse?* We don't know, but it doesn't matter.
Q: Shouldn't the warehouse be located at a factory to save at least one trip? It could be a solution but the other trips will be longer so it's not more economical.	*Q: What does the most economical mean?* It is necessary to pay the lowest price, we will have to take into account the fuel consumption, the distances covered,...	*Q: How many kilometers can a truck drive with a full tank of gas and where are the gas stations?* It can be 2000km long and the gas stations are every 500km.	*Q: Are there many radars?* Will this help us to position the warehouse?

Based on the questions and answers of the students, we designed the "relaunched realistic fiction". In this particular case, it was decided to take into account only distances in the cost of the transportation "as the crow flies", to consider only one truck which can be not full and deliver to various factories in one trip, to allow a factory to be delivered to in many trips, and that a factory can store some supplies in advance. Based on these decisions, the classes were asked to find an optimal solution for this (fully mathematized) problem.

Figure 4. Example from students' work.

stratégie1				
Nombre de livraisons.				
Semaine	1	2	3	4
Echo(240)	2	2	2	2
Tango(80)	1	1	0	1
Foxtrot(150)	2	1	1	1
November(150	2	1	1	1
Quantité de marchandises restante après livraison.				
Semaine	1	2	3	4
Echo(240)	120+120 - 240=0	120+120 - 240=0	120+120 - 240=0	120+120 - 240=0
Tango(80)	120-80=40	120+40-80=80	0	120-80=40
Foxtrot(150)	120+120-150=90	120+90-150 = 60	120+60-150=30	120+30-150=0
November(150	120+120-150 = 90	120+90-150=60	120+60-150=30	120+30-150=0

Figure 5. Example from students’ work.

Figures 4 and 5 show examples of students’ work based on the "Warehouse" problem. During the fourth week, several classes used the spreadsheet or Geogebra to solve the problem. In the first example, the students chose to go back and forth between the warehouse and the factories, considering the possibility of storage.
In the second example, the students chose a location close to Echo (because it has the greatest need), while being at equal distance from Foxtrot and November (same needs), and chose an order for the trips according to needs, also allowing themselves to store materials. Their strategy is based on having as little as possible left in the warehouse, and on moving the goods left the week before.
At the end of the activity, the students shared their solutions on the online forum and an “advanced solution” was distributed.

Conclusions and perspectives

In this work, we developed an epistemological analysis of experts’ practice of modeling, which allowed us to identify invariant practices in modeling (in different contexts) and also practices that seems to be context-specific.
This work permits us to design problems that foster modeling activity, and particularly horizontal mathematization. These problems are designed following specific characteristics of situations called “realistic fictions”, more precisely as realistic fictions conceived as adaptations of professional modeling practices. These situations are implemented and experimented in the RESCO device, that allows us to make tens of classes work collaboratively through a forum.
Our experiments show a devolution of the modeling necessity at stake, and the strong involvement of students. The epistemological analysis also helped us to develop indicators that permit us to compare students’ modeling practices to experts’, and to discuss the way they take into account the context of the situation in order to question its modeling.
This work is still ongoing, and many perspectives have yet to be developed. First, we want to go on with the epistemological study, and will conduct more interviews with experts in industrial problems. Second, as we have shown that there is an articulation between generic modeling practices and context-specific practices, we want to deepen the understanding of this articulation. This can be done with an analysis of the previous interviews, but also by exploring the students’ work in this direction. This would permit us to better take into account the context while designing task, making *a priori* analyses of modeling tasks, and studying students’ productions, in particular concerning the question-and-answer phases in our device.

References

Brousseau G., Warfield V. (2014) Didactic Situations in Mathematics Education. In: Lerman S. (eds) *Encyclopedia of Mathematics Education*. Springer, Dordrecht

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Kluwer Academic.

Israël, G. (1996). *La mathématisation du réel: essai sur la modélisation mathématique*. Seuil Eds.

Treffers, A. (1978) *Wiskobas Doelgericht*. Utrecht: IOWO.

Yvain, S. (2018) Favoriser la dévolution de la mathématisation horizontale aux élèves engagés dans une activité de modélisation in *Proceedings of the CIEAEM 69 "Quaderni di Ricerca in Didattica (Mathematics)"*, n. 27, Supplemento n.2, 2017 (pp. 123–128).

Yvain, S. & Modeste, S. (2018) Faire entrer les élèves dans la mathématisation horizontale. Des fictions réalistes et un dispositif de résolution collaborative in *Proceedings of the CIEAEM 69 "Quaderni di Ricerca in Didattica (Mathematics)"*, n. 27, Supplemento n.2, 2017 (pp. 288–296).

Yvain-Prébiski, S. (2018). *Étude de la transposition à la classe de pratiques de chercheurs en modélisation mathématique dans les sciences du vivant. Analyse des conditions de la dévolution de la mathématisation horizontale aux élèves,* Doctoral dissertation, Université de Montpellier.